PH208: Introduction to Boltzmann Transport Equation

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Contents

1 Introduction

1.1 Distribution Function and Transport

Transport refers to the flow of currents in response to applied fields. These include:

- Electrical current j
- Thermal current j_Q

The driving forces are:

- Electric field E
- Temperature gradient ∇T

We assume the system is always close to equilibrium, so responses are linear. The central object is the distribution function:

$$f(x, k, t) \tag{1}$$

which gives the probability of finding an electron near (x, k) at time t.

At equilibrium:

$$f_0(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1} \tag{2}$$

A key identity:

$$\frac{df_0}{d\varepsilon} = -\frac{1}{k_B T} \frac{e^{\beta(\varepsilon-\mu)}}{(e^{\beta(\varepsilon-\mu)} + 1)^2} \tag{3}$$

At low temperature:

$$-\frac{df_0}{d\varepsilon} \to \delta(\varepsilon - E_F) \tag{4}$$

Thus, transport is governed by states near E_F. More precisely, transport is governed by states within an energy window of order $k_B T$ around E_F.

1.2 Semiclassical Dynamics

Electron motion in a band:

$$\varepsilon = \varepsilon(k) \tag{5}$$

Velocity:

$$v_k = \frac{1}{\hbar} \frac{d\varepsilon}{dk} \tag{6}$$

Equation of motion:

$$\hbar \frac{dk}{dt} = F \tag{7}$$

For electric field:

$$F = -eE \tag{8}$$

Effective Mass Tensor In general band structures, the response to forces is not isotropic. The effective mass tensor governs the acceleration:

$$M_{ij}^{-1} = \frac{1}{\hbar^2} \frac{\partial^2 \varepsilon}{\partial k_i \partial k_j}. \tag{9}$$

This generalizes the concept of mass to anisotropic bands and becomes essential for describing transport in real crystals. This means the acceleration of an electron **a** and the applied force **F** are generally not parallel. This is critical for understanding transport in low-symmetry crystals.

2 Phase Space and Boltzmann Equation

2.1 What leads to non-equilibrium distribution?

The distribution function $f(\boldsymbol{r}, \boldsymbol{k}, t)$ deviates from the equilibrium distribution f_0 because the system is subjected to **external perturbations** that "push" particles out of their statistically most probable, uniform state.

In equilibrium, the forces acting on the system are perfectly balanced, the chemical potential and temperature are uniform, and scattering events exactly cancel each other out. When you apply an external influence, this balance is broken.

The deviation occurs due to the following factors:

External Driving Forces (Drift)

The Boltzmann Transport Equation (BTE) contains terms that represent the motion of particles through phase space due to external fields. These are the "driving" terms:

- **Electric Fields ($\boldsymbol{E}$):** An electric field exerts a force on charged particles (electrons), accelerating them. This changes their momentum ($\boldsymbol{k}$), causing the distribution to shift in momentum space.
- **Thermal Gradients (∇T):** If one end of a sample is hotter than the other, particles on the hot side have higher average energy. This creates a natural tendency for particles to diffuse toward the colder region, shifting the distribution in both position ($\boldsymbol{r}$) and momentum ($\boldsymbol{k}$) space.
- **Chemical Potential Gradients ($\nabla \mu$):** Differences in particle concentration (or electrochemical potential) drive particles to flow from regions of high concentration to low concentration.

These terms act as a "wind" that constantly drags the distribution function away from the symmetric, stationary equilibrium state.

The Competition: Drift vs. Scattering

The reason the distribution function does not grow infinitely away from equilibrium is the **collision integral** ($I_{\boldsymbol{k}}\{f\}$).

- **Scattering as a Restoring Force:** Collisions (with impurities, phonons, or other electrons) act to randomize the momentum of the particles. Every time an electron scatters, it "forgets" some of the acceleration it received from the external field.
- **Steady State:** In a steady-state system, the distribution function stops changing ($\frac{\partial f}{\partial t} = 0$). This happens because the **drift** caused by external fields is exactly balanced by the **redistribution** caused by collisions.

Summary

You can think of the equilibrium distribution (f_0) as a **calm, still pond**.

- **External fields** act like a paddle, creating a current in that pond (the drift).
- **Collisions** act like the viscosity of the water, resisting the current and trying to bring the pond back to a still state.

2.2 Evolution of distribution function

The deviation ($\delta f = f - f_0$) is the net result of this tug-of-war. It is the "ripple" in the distribution function that allows for the transport of charge and heat, which we then measure as electrical current or heat flow.

The distribution evolves in the phase space (x, k). Total derivative:

$$\frac{df(x,k,t)}{dt} = \frac{\partial f}{\partial t} + \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dk}{dt}\frac{\partial f}{\partial k} = \frac{\partial f}{\partial t} + v_k\frac{\partial f}{\partial x} + \frac{F}{\hbar}\frac{\partial f}{\partial k} \tag{10}$$

In the absence of collisions:

$$\frac{df}{dt} = 0 \tag{11}$$

Thus:

$$\frac{\partial f}{\partial t} + v_k\frac{\partial f}{\partial x} + \frac{F}{\hbar}\frac{\partial f}{\partial k} = 0 \tag{12}$$

This expresses conservation of particles in phase space. This result follows from the continuity equation in phase space:

$$\frac{\partial f}{\partial t} + \nabla \cdot (\mathbf{u}f) = 0 \tag{13}$$

where $\mathbf{u} = (\dot{x}, \dot{k})$ and $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial k})$. In 3-D, these are six-dimensional vectors. Because phase-space flow is incompressible ($\nabla \cdot u = 0$), the distribution function is conserved along trajectories. This is a statement of Liouville's theorem and forms the foundation of the Boltzmann equation.

$$\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f = 0 \tag{14}$$

3 Inclusion of Collisions

In reality, collisions modify the distribution function. We therefore write the full Boltzmann equation as:

$$\frac{\partial f}{\partial t} + v_k \frac{\partial f}{\partial x} + \frac{F}{\hbar}\frac{\partial f}{\partial k} = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \equiv I_k\{f\} \tag{15}$$

The quantity $I_k\{f\}$ is called the **collision integral**. It represents the rate of change of the distribution function at momentum k due to scattering processes.

Physical meaning:

$$I_k\{f\} = (\text{rate of scattering into state } k) - (\text{rate of scattering out of state } k) \tag{16}$$

Thus, it describes the net flow of probability in momentum space.

3.1 Microscopic Collision Integral

The microscopic form of the collision integral is:

$$I_k\{f\} = \sum_{k'} \left[W_{k'\to k} f(k')(1 - f(k)) - W_{k\to k'} f(k)(1 - f(k'))\right] \tag{17}$$

where $W_{k\to k'}$ is the transition rate from k to k'.

Interpretation of terms:

- $W_{k'\to k} f(k')(1 - f(k))$: scattering into state k
- $W_{k\to k'} f(k)(1 - f(k'))$: scattering out of state k
- $(1 - f)$ factors encode Pauli blocking.

3.2 Origin of Transition Rates

The transition rates are determined by microscopic physics, typically via Fermi's Golden Rule:

$$W_{k\to k'} \propto |\langle k'|U|k\rangle|^2 \delta(\varepsilon_k - \varepsilon_{k'}) \tag{18}$$

where U is the scattering potential (e.g., impurities, phonons).

Thus, $I_k\{f\}$ contains the full microscopic information about scattering.

3.3 Functional Nature of $I_k\{f\}$

The notation $\{f\}$ emphasizes that:

$$I_k\{f\} \text{ depends on } f(k) \text{ and also on } f(k') \text{ for all } k'. \tag{19}$$

Hence, it is not a simple function but a **functional** of the entire distribution.

3.4 General Properties

1. Particle conservation:

$$\int dk\, I_k\{f\} = 0 \tag{20}$$

Collisions do not change the total number of particles.

2. Collisional invariants:

For any conserved quantity $\mathcal{F}(k)$,

$$\int dk\, \mathcal{F}(k)\, I_k\{f\} = 0 \tag{21}$$

Examples:

- $\mathcal{F} = 1$: particle number
- $\mathcal{F} = \varepsilon(k)$: energy

3. Equilibrium condition:

$$I_k\{f_0\} = 0 \tag{22}$$

The equilibrium distribution is a fixed point of the collision dynamics.

3.5 Physical Picture

The collision integral describes how scattering redistributes electrons:

- electrons arrive into k from other states
- electrons leave k to other states

Thus, transport can be viewed as a competition between:

- drift in phase space (left-hand side)
- redistribution via collisions (right-hand side)

3.6 Why We Approximate $I_k\{f\}$

The exact collision integral is:

- nonlinear in f
- coupled across all k
- difficult to solve analytically

Therefore, we replace it with a simpler form in the relaxation time approximation.

4 Local Equilibrium and Linearization

Collisions act on short time scales and tend to establish a **local equilibrium** distribution:

$$f_0(x,k) = \frac{1}{e^{(\varepsilon(k)-\mu(x))/k_B T(x)} + 1} \tag{23}$$

Here, both the chemical potential $\mu(x)$ and temperature $T(x)$ are allowed to vary slowly in space.

However, $f_0(x,k)$ is *not* an exact solution of the Boltzmann equation because spatial gradients generate drift in phase space.

We therefore write:

$$f(x,k,t) = f_0(x,k) + \delta f(x,k,t) \tag{24}$$

where δf is a small deviation.

4.1 Derivatives of the Local Equilibrium Distribution

We now compute the derivatives of f_0, which are needed in the Boltzmann equation.

Since $f_0 = f_0(\varepsilon, \mu(x), T(x))$, we compute its total differential:

$$df_0 = \frac{\partial f_0}{\partial \varepsilon} d\varepsilon + \frac{\partial f_0}{\partial \mu} d\mu + \frac{\partial f_0}{\partial T} dT \tag{25}$$

4.2 Useful identities

From the definition of the Fermi function:

$$\frac{\partial f_0}{\partial \mu} = -\frac{\partial f_0}{\partial \varepsilon} \tag{26}$$

$$\frac{\partial f_0}{\partial T} = \frac{\varepsilon - \mu}{T} \frac{\partial f_0}{\partial \varepsilon} \tag{27}$$

4.3 Spatial derivative of f_0

Using

$$d\mu = \frac{\partial \mu}{\partial x} dx, \quad dT = \frac{\partial T}{\partial x} dx, \tag{28}$$

we obtain:

$$\frac{\partial f_0}{\partial x} = \frac{\partial f_0}{\partial \mu} \frac{\partial \mu}{\partial x} + \frac{\partial f_0}{\partial T} \frac{\partial T}{\partial x} \tag{29}$$

Substituting:

$$\frac{\partial f_0}{\partial x} = \left(-\frac{\partial f_0}{\partial \varepsilon} \right) \left[\frac{\partial \mu}{\partial x} + \frac{\varepsilon - \mu}{T} \frac{\partial T}{\partial x} \right] \tag{30}$$

4.4 Momentum derivative of f_0

Since f_0 depends on k only through $\varepsilon(k)$:

$$\frac{\partial f_0}{\partial k} = \frac{df_0}{d\varepsilon} \frac{d\varepsilon}{dk} \tag{31}$$

Using

$$\frac{d\varepsilon}{dk} = \hbar v_k \tag{32}$$

we obtain:

$$\frac{\partial f_0}{\partial k} = \hbar v_k \frac{\partial f_0}{\partial \varepsilon} \tag{33}$$

4.5 Substitution into the Boltzmann Equation

We now substitute:

$$f = f_0 + \delta f \tag{34}$$

into:

$$\frac{\partial f}{\partial t} + v_k \frac{\partial f}{\partial x} + \frac{F}{\hbar} \frac{\partial f}{\partial k} = I_k\{f\} \tag{35}$$

to get:

$$\frac{\partial \delta f}{\partial t} + v_k \frac{\partial \delta f}{\partial x} + \frac{F}{\hbar} \frac{\partial \delta f}{\partial k} + v_k \frac{\partial f_0}{\partial x} + \frac{F}{\hbar} \frac{\partial f_0}{\partial k} = I_k\{f_0 + \delta f\} \tag{36}$$

4.6 Evaluate the driving terms

The two driving terms are (see Eqn. 30 and Eqn. 33):

$$v_k \frac{\partial f_0}{\partial x} = v_k \left(-\frac{\partial f_0}{\partial \varepsilon} \right) \left[\frac{\partial \mu}{\partial x} + \frac{\varepsilon - \mu}{T} \frac{\partial T}{\partial x} \right] \tag{37}$$

$$\frac{F}{\hbar}\frac{\partial f_0}{\partial k} = F v_k \frac{\partial f_0}{\partial \varepsilon} \tag{38}$$

With $F = -eE$, this becomes:

$$\frac{F}{\hbar}\frac{\partial f_0}{\partial k} = -eE v_k \frac{\partial f_0}{\partial \varepsilon} \tag{39}$$

The net driving term is:

$$v_k \frac{\partial f_0}{\partial x} + \frac{F}{\hbar}\frac{\partial f_0}{\partial k} = v_k \left(-\frac{\partial f_0}{\partial \varepsilon}\right)\left[eE + \frac{\partial \mu}{\partial x} + \frac{\varepsilon - \mu}{T}\frac{\partial T}{\partial x}\right] \tag{40}$$

Electrochemical field It is useful to define an effective field:

$$E^* = E + \frac{1}{e}\frac{\partial \mu}{\partial x}. \tag{41}$$

Then the driving term becomes:

$$v_k \left[eE^* + \frac{\varepsilon - \mu}{T}\frac{\partial T}{\partial x}\right]\left(-\frac{\partial f_0}{\partial \varepsilon}\right). \tag{42}$$

This shows that transport is driven by gradients of the electrochemical potential. Combining all the terms, we get the final **Linearized Boltzmann equation (LBE)**:

$$\frac{\partial \delta f}{\partial t} + v_k \frac{\partial \delta f}{\partial x} + v_k \left[eE^* + \frac{\varepsilon - \mu}{T}\frac{\partial T}{\partial x}\right]\left(-\frac{\partial f_0}{\partial \varepsilon}\right) = I_k\{\delta f\} \tag{43}$$

Note that we have used

$$I_k\{f_0 + \delta f\} \approx I_k\{\delta f\} \tag{44}$$

because:

$$I_k\{f_0\} = 0 \tag{45}$$

and we keep only first-order terms in δf. Also, all terms that are products of two small quantities (like $F \cdot \delta f$) are neglected. This is the **Linear Response** approximation.

4.7 Physical interpretation

The equation has three components:

- Drift of δf in phase space
- Driving terms from:
  - electric field E
  - temperature gradient ∇T
- Relaxation via collisions (right-hand side)

Thus, δf is generated by external fields and opposed by scattering. In other words:

- external fields distort f
- collisions restore equilibrium
- steady state = balance of the two

5 Relaxation Time Approximation

The exact form of $I_k\{\delta f\}$ is unknown. To make progress, we use an approach called **phenomenological closure**. We replace the exact collision integral by a simpler phenomenological model, in this case, a relaxation time approximation. The microscopic scattering is not explicitly computed. Explicitly, we replace the collision integral by:

$$I_k\{\delta f\} = -\frac{f - f_0}{\tau} \tag{46}$$

This assumes that in the absence of any fields or temperature and electrochemical potential gradients, the Boltzmann equation becomes:

$$\delta f(t) = \delta f(0) e^{-t/\tau} \tag{47}$$

The distribution thereby relaxes to the equilibrium one on the scale of τ.

Dominant scattering mechanisms The relaxation time depends on the microscopic scattering processes:

- Impurity scattering: $\tau \sim$ const (dominates at low temperature)
- Phonon scattering: $\tau^{-1} \propto T$ (high temperature regime)
- Electron-electron scattering: $\tau^{-1} \propto T^2$ (Fermi liquid behavior)

The net scattering time is determined by Matthiessen's Rule:

$$\frac{1}{\tau_{total}} = \frac{1}{\tau_{imp}} + \frac{1}{\tau_{ph}} + \frac{1}{\tau_{ee}}$$

6 Electrical and Thermoelectric Transport

This section details the derivation of transport properties by solving the Boltzmann equation in the presence of an electric field E and a temperature gradient ∇T.

6.1 Solution for δf

The Linear Boltzmann Equation (LBE) under the relaxation time approximation

$$\frac{\partial \delta f}{\partial t} + v_k \frac{\partial \delta f}{\partial x} + v_k \left[eE^* + \frac{\varepsilon - \mu}{T} \frac{\partial T}{\partial x} \right] \left(-\frac{\partial f_0}{\partial \varepsilon} \right) = I_k\{\delta f\} = -\frac{f - f_0}{\tau} \tag{48}$$

is used to find the deviation from equilibrium, $\delta f = f - f_0$.

Assuming a steady state ($\frac{\partial \delta f}{\partial t} = 0$) and spatial uniformity of δf ($\frac{\partial \delta f}{\partial x} = 0$), the equation simplifies to:

$$v_k \left[eE^* + \frac{\epsilon - \mu}{T} \frac{\partial T}{\partial x} \right] \left(-\frac{\partial f_0}{\partial \epsilon} \right) = -\frac{\delta f}{\tau} \tag{49}$$

Where E^* is the **electrochemical field**, defined as $E^* = E + \frac{1}{e} \frac{\partial \mu}{\partial x}$. Solving for δf gives:

$$\delta f = \tau v_k \left(-\frac{\partial f_0}{\partial \epsilon} \right) \left[eE^* + \frac{\epsilon - \mu}{T} \frac{\partial T}{\partial x} \right] \tag{50}$$

6.2 Electrical Current

The electrical current density j is defined by the integral of the velocity over the occupied states (in 1D, for simplicity):

$$j = -e \int \frac{dk}{2\pi} v_k f(k) \tag{51}$$

Since the equilibrium distribution f_0 carries no net current, the expression reduces to the contribution from δf:

$$j = -e \int \frac{dk}{2\pi} v_k \delta f \tag{52}$$

Substituting the expression for δf derived above yields:

$$j = e \int \frac{dk}{2\pi} v_k^2 \tau \left(-\frac{\partial f_0}{\partial \epsilon} \right) \left[eE^* + \frac{\epsilon - \mu}{T} \frac{\partial T}{\partial x} \right] \tag{53}$$

6.3 Separation of Contributions and General Transport Equation

The total current j is a sum of the response to the electrochemical field and the response to the temperature gradient:

$$j = j_{E^*} + j_T \tag{54}$$

We define the transport coefficients as:

- **Electrical Conductivity (σ)**:

$$\sigma = e^2 \int \frac{dk}{2\pi} v_k^2 \tau \left(-\frac{\partial f_0}{\partial \epsilon} \right) \tag{55}$$

- **Thermoelectric Response (α)**:

$$\alpha = \frac{e}{T} \int \frac{dk}{2\pi} v_k^2 \tau (\epsilon - \mu) \left(-\frac{\partial f_0}{\partial \epsilon} \right) \tag{56}$$

The **General Transport Equation** is written as:

$$j = \sigma E^* - \alpha \frac{\partial T}{\partial x} \tag{57}$$

6.4 Energy Representation and Key Results

Using the transport distribution function $\Sigma(\epsilon)$ in 3D (this is a second-rank tensor):
Using the density of states:

$$g(\varepsilon) = \frac{1}{2\pi\hbar |v_k|} \tag{58}$$

we define:

$$\Sigma(\varepsilon) = \int \frac{d\mathbf{k}}{4\pi^3} \, \mathbf{v_k} \mathbf{v_k} \, \tau(\varepsilon) \, \delta(\varepsilon - \varepsilon_{\mathbf{k}}) \tag{59}$$

Physical interpretation The transport distribution function $\Sigma(\varepsilon)$ encodes:

- band structure via $\mathbf{v_k}$
- scattering via $\tau(\varepsilon)$
- density of states via phase-space integration

Thus, all transport coefficients are determined by the behavior of $\Sigma(\varepsilon)$ near the Fermi energy. The transport coefficients are determined by the behavior of $\Sigma(\epsilon)$ near the Fermi energy:

$$\sigma = e^2 \int d\epsilon \Sigma(\epsilon) \left(-\frac{\partial f_0}{\partial \epsilon} \right) \tag{60}$$

$$\alpha = \frac{e}{T} \int d\epsilon \Sigma(\epsilon)(\epsilon - \mu) \left(-\frac{\partial f_0}{\partial \epsilon} \right) \tag{61}$$

The **Seebeck Coefficient** S is defined under open-circuit conditions ($j = 0$) as $S = \frac{\alpha}{\sigma}$. Explicitly:

$$S = \frac{1}{eT} \frac{\int d\varepsilon \, \Sigma(\varepsilon)(\varepsilon - \mu) \left(-\frac{\partial f_0}{\partial \varepsilon} \right)}{\int d\varepsilon \, \Sigma(\varepsilon) \left(-\frac{\partial f_0}{\partial \varepsilon} \right)} \tag{62}$$

In the low-temperature limit (using Sommerfeld expansion), this yields the **Mott Formula**:

$$S = \frac{\pi^2 k_B^2 T}{3e} \left. \frac{d}{d\epsilon} \ln \Sigma(\epsilon) \right|_{\epsilon = E_F} \tag{63}$$

The relaxation time is often energy dependent. $\tau(\epsilon)$ often follows power laws (e.g., $\tau \propto \epsilon^{3/2}$ for ionized impurity scattering). This energy dependence is what ultimately drives a non-zero Seebeck coefficient in the Mott Formula.

- **The Seebeck Coefficient (S) as a Probe of Asymmetry:** The Seebeck coefficient is a direct measure of the **particle-hole asymmetry** of the transport distribution function $\Sigma(\epsilon)$ near the Fermi level. Physically, a temperature gradient drives both high-energy "hot" carriers and low-energy "cold" carriers in the same direction.
- **Cancellation in Symmetric Bands:** If the band structure and scattering mechanisms are perfectly symmetric around the Fermi energy (i.e., $\Sigma(\epsilon)$ is flat at E_F), the electron-like contributions from states above μ and the hole-like contributions from states below μ generate equal and opposite currents that exactly cancel. In such a case, the net thermal voltage is zero ($S = 0$).
- **The Mott Insight:** As captured by the Mott formula, $S \propto \frac{d}{d\epsilon} \ln \Sigma(\epsilon)|_{E_F}$, a non-zero Seebeck effect only arises when there is a gradient in the material's ability to transport charge across the Fermi surface. This makes S an incredibly sensitive tool for detecting subtle changes in the density of states or energy-dependent scattering rates that conductivity (σ) alone might mask.

Finally, the **Wiedemann-Franz Law** relates thermal and electrical conductivity:

$$\frac{\kappa}{\sigma T} = \frac{\pi^2}{3} \left(\frac{k_B}{e} \right)^2 \tag{64}$$

This reflects that the same quasiparticles carry both heat and charge.

7 Magnetic Field and Hall Conductivity

We now include a magnetic field within the semiclassical Boltzmann framework. We begin with the general Boltzmann equation in phase space:

$$\frac{\partial f}{\partial t} + \frac{d\mathbf{r}}{dt} \cdot \nabla_{\mathbf{r}} f + \frac{d\mathbf{k}}{dt} \cdot \nabla_{\mathbf{k}} f = I_k\{f\} \tag{65}$$

The semiclassical equations of motion give:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}_k, \qquad \frac{d\mathbf{k}}{dt} = -\frac{e}{\hbar} \left(\mathbf{E} + \mathbf{v}_k \times \mathbf{B} \right). \tag{66}$$

Substituting into the Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}_k \cdot \nabla_{\mathbf{r}} f - \frac{e}{\hbar} \left(\mathbf{E} + \mathbf{v}_k \times \mathbf{B} \right) \cdot \nabla_{\mathbf{k}} f = I_k\{f\} \tag{67}$$

7.1 Steady-State and Uniform Limit

For transport, we assume:

$$\frac{\partial f}{\partial t} = 0, \qquad \nabla_{\mathbf{r}} f = 0, \tag{68}$$

which gives:

$$-\frac{e}{\hbar}\left(\mathbf{E} + \mathbf{v}_k \times \mathbf{B}\right) \cdot \nabla_{\mathbf{k}} f = I_k\{f\}. \tag{69}$$

Using the relaxation time approximation,

$$I_k\{f\} = -\frac{f - f_0}{\tau}, \tag{70}$$

we obtain:

$$\boxed{-\frac{e}{\hbar}\left(\mathbf{E} + \mathbf{v}_k \times \mathbf{B}\right) \cdot \nabla_{\mathbf{k}} f = -\frac{f - f_0}{\tau}} \tag{71}$$

7.2 Systematic Expansion in Fields

We expand the distribution function around the equilibrium:

$$f = f_0 + \delta f, \qquad \delta f = \delta f^{(0)} + \delta f^{(1)} + \cdots, \tag{72}$$

where:

- $\delta f^{(0)} \sim E$ (longitudinal response),
- $\delta f^{(1)} \sim EB$ (Hall response).

Substituting into the Boltzmann equation:

$$-\frac{e}{\hbar}\left(\mathbf{E} + \mathbf{v}_k \times \mathbf{B}\right) \cdot \nabla_{\mathbf{k}}(f_0 + \delta f) = -\frac{\delta f}{\tau}. \tag{73}$$

Expanding:

$$-\frac{e}{\hbar}\left(\mathbf{E} + \mathbf{v}_k \times \mathbf{B}\right) \cdot \nabla_{\mathbf{k}} f_0 - \frac{e}{\hbar}\left(\mathbf{E} + \mathbf{v}_k \times \mathbf{B}\right) \cdot \nabla_{\mathbf{k}} \delta f = -\frac{\delta f}{\tau}. \tag{74}$$

7.3 Zeroth-Order Solution

At leading order, we retain terms linear in $\mathbf{E}$:

$$-\frac{e}{\hbar}\mathbf{E} \cdot \nabla_{\mathbf{k}} f_0 = -\frac{\delta f^{(0)}}{\tau}. \tag{75}$$

Using:

$$\nabla_{\mathbf{k}} f_0 = \hbar \mathbf{v}_k \frac{\partial f_0}{\partial \varepsilon}, \tag{76}$$

we obtain:

$$\boxed{\delta f^{(0)} = e\tau \left(-\frac{\partial f_0}{\partial \varepsilon} \right) \mathbf{v}_k \cdot \mathbf{E}.} \tag{77}$$

The magnetic field does not contribute at this order, since:

$$(\mathbf{v}_k \times \mathbf{B}) \cdot \nabla_{\mathbf{k}} f_0 = \hbar (\mathbf{v}_k \times \mathbf{B}) \cdot \mathbf{v}_k \frac{\partial f_0}{\partial \varepsilon} = 0. \tag{78}$$

7.4 First-Order Correction (Hall Term)

At next order, we retain terms proportional to EB. The Boltzmann equation gives:

$$-\frac{e}{\hbar} \mathbf{E} \cdot \nabla_{\mathbf{k}} \delta f^{(0)} - \frac{e}{\hbar} (\mathbf{v}_k \times \mathbf{B}) \cdot \nabla_{\mathbf{k}} f_0 - \frac{e}{\hbar} (\mathbf{v}_k \times \mathbf{B}) \cdot \nabla_{\mathbf{k}} \delta f^{(0)} = -\frac{\delta f^{(1)}}{\tau}. \tag{79}$$

The second term vanishes identically, while the first term contributes at order E^2 and is neglected in linear response. Thus:

$$\boxed{\delta f^{(1)} = \frac{e\tau}{\hbar} (\mathbf{v}_k \times \mathbf{B}) \cdot \nabla_{\mathbf{k}} \delta f^{(0)}.} \tag{80}$$

7.5 Evaluating $\nabla_{\mathbf{k}} \delta f^{(0)}$

From:

$$\delta f^{(0)} = e\tau \left(-\frac{\partial f_0}{\partial \varepsilon} \right) (\mathbf{v}_k \cdot \mathbf{E}), \tag{81}$$

we compute:

$$\nabla_{\mathbf{k}} \delta f^{(0)} = e\tau \left[(\nabla_{\mathbf{k}} (\mathbf{v}_k \cdot \mathbf{E})) \left(-\frac{\partial f_0}{\partial \varepsilon} \right) + (\mathbf{v}_k \cdot \mathbf{E}) \nabla_{\mathbf{k}} \left(-\frac{\partial f_0}{\partial \varepsilon} \right) \right]. \tag{82}$$

The second term vanishes upon integration by symmetry. Thus:

$$\nabla_{\mathbf{k}} \delta f^{(0)} \approx e\tau \left(-\frac{\partial f_0}{\partial \varepsilon} \right) \nabla_{\mathbf{k}} (\mathbf{v}_k \cdot \mathbf{E}). \tag{83}$$

For a parabolic band:

$$\mathbf{v}_k = \frac{\hbar \mathbf{k}}{m} \quad \Rightarrow \quad \nabla_{\mathbf{k}}(\mathbf{v}_k \cdot \mathbf{E}) = \frac{\hbar}{m}\mathbf{E}. \tag{84}$$

Thus:

$$\boxed{\nabla_{\mathbf{k}} \delta f^{(0)} = e\tau \frac{\hbar}{m} \left(-\frac{\partial f_0}{\partial \varepsilon} \right) \mathbf{E}.} \tag{85}$$

7.6 Hall Correction to Distribution

Substituting:

$$\boxed{\delta f^{(1)} = e^2 \tau^2 \frac{1}{m} \left(-\frac{\partial f_0}{\partial \varepsilon} \right) (\mathbf{v}_k \times \mathbf{B}) \cdot \mathbf{E}.} \tag{86}$$

7.7 Hall Current

$$\mathbf{j}^{(H)} = -e \int \frac{d^d k}{(2\pi)^d} \mathbf{v}_k \delta f^{(1)}, \tag{87}$$

$$\mathbf{j}^{(H)} = -e^3 \tau^2 \frac{1}{m} \int \frac{d^d k}{(2\pi)^d} \mathbf{v}_k \left(-\frac{\partial f_0}{\partial \varepsilon} \right) (\mathbf{v}_k \times \mathbf{B}) \cdot \mathbf{E}. \tag{88}$$

7.8 Tensor Structure

Using:

$$(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{E} = \mathbf{v} \cdot (\mathbf{E} \times \mathbf{B}), \tag{89}$$

and isotropy:

$$\langle v_i v_j \rangle = \frac{v_F^2}{d} \delta_{ij}, \tag{90}$$

we obtain:

$$\boxed{\mathbf{j}^{(H)} = \sigma(\omega_c \tau)(\mathbf{E} \times \hat{z}).} \tag{91}$$

7.9 Hall Conductivity

$$\boxed{\sigma_{xy} = \sigma\, \omega_c \tau, \qquad \sigma_{xx} = \frac{n e^2 \tau}{m},} \tag{92}$$

where:

$$\omega_c = \frac{eB}{m}. \tag{93}$$

7.10 Hall Coefficient

$$\boxed{R_H = \frac{1}{ne}.} \tag{94}$$

7.11 Physical Picture

- The electric field distorts the Fermi surface.
- The magnetic field rotates this distortion in momentum space.
- The resulting transverse current gives rise to the Hall effect.
- The response appears at order EB.

8 Berry Curvature and Anomalous Transport

8.1 Semiclassical Dynamics with Berry Curvature

In crystals with broken time-reversal or inversion symmetry, the semiclassical equations of motion acquire an additional term due to Berry curvature.

The modified equations of motion are:

$$\dot{\mathbf{r}} = \mathbf{v}_k + \frac{e}{\hbar}\mathbf{E} \times \boldsymbol{\Omega}(\mathbf{k})$$

$$\hbar\dot{\mathbf{k}} = -e\left(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}\right)$$

where:

- $\boldsymbol{\Omega}(\mathbf{k})$ is the Berry curvature
- $\mathbf{v}_k = \frac{1}{\hbar}\nabla_k \varepsilon_k$

8.2 Definition of Berry Curvature

The Berry curvature is defined as:

$$\boldsymbol{\Omega}_n(\mathbf{k}) = \nabla_k \times \mathbf{A}_n(\mathbf{k})$$

where the Berry connection is:

$$\mathbf{A}_n(\mathbf{k}) = i\langle u_{n\mathbf{k}} | \nabla_k | u_{n\mathbf{k}} \rangle$$

Here $|u_{n\mathbf{k}}\rangle$ is the periodic part of the Bloch wavefunction.

Symmetries constrain the Berry curvature: $\mathbf{\Omega}(\mathbf{k})$ is an odd function of $\mathbf{k}$ under time-reversal symmetry (TRS) and an even function under inversion symmetry (IS). For $\mathbf{\Omega}(\mathbf{k})$ to contribute to net transport, at least one of these symmetries must be broken.

8.3 Physical Interpretation

Berry curvature acts as an effective magnetic field in momentum space, functioning as a "momentum-space monopole":

- It modifies electron trajectories by exerting a pseudo-Lorentz force.
- It leads to transverse motion even without an external magnetic field.

8.4 Anomalous Velocity

The additional term:

$$\mathbf{v}_{\text{an}} = \frac{e}{\hbar}\mathbf{E} \times \mathbf{\Omega}(\mathbf{k})$$

is called the anomalous velocity. This term is perpendicular to both $\mathbf{E}$ and $\mathbf{\Omega}$, leading to a transverse current.

8.5 Intrinsic Anomalous Hall Effect

The intrinsic anomalous Hall current results from the band structure itself:

$$\mathbf{j}_{\text{AH}} = -e \int \frac{d^d k}{(2\pi)^d} \left(\frac{e}{\hbar}\mathbf{E} \times \mathbf{\Omega}(\mathbf{k}) \right) f_0(\varepsilon_k)$$

$$\mathbf{j}_{\text{AH}} = \frac{e^2}{\hbar}\mathbf{E} \times \int \frac{d^d k}{(2\pi)^d} \mathbf{\Omega}(\mathbf{k}) f_0(\varepsilon_k)$$

Thus, the Hall conductivity is:

$$\sigma_{xy} = \frac{e^2}{\hbar} \int \frac{d^d k}{(2\pi)^d} \Omega_z(\mathbf{k}) f_0(\varepsilon_k)$$

Note that real materials may also include extrinsic contributions such as skew scattering or side jump.

8.6 Key Insight

- Hall response can arise without a magnetic field.

- It is an intrinsic property of the band structure.
- It depends on the Berry curvature of occupied states.

8.7 Topological Interpretation

For 2D insulators, the Hall conductivity is related to the global topological properties of the band:

$$\sigma_{xy} = \frac{e^2}{h} C$$

where C is the Chern number:

$$C = \frac{1}{2\pi} \int_{\mathrm{BZ}} d^2k \, \Omega_z(\mathbf{k})$$

This quantization, representing the integral of the Berry curvature over the entire Brillouin zone, underlies the quantum Hall effect.

9 Hydrodynamic Transport Regime

9.1 Transport Regimes

Electron transport can be classified based on the hierarchy of scattering length scales:

- **Ballistic regime:** $l \gg L$.
- **Diffusive regime:** $l \ll L$.
- **Hydrodynamic regime:** $l_{ee} \ll W \ll l_{\mathrm{imp}}, L$.

where l is the mean free path, l_{ee} is the electron-electron scattering length, l_{imp} is the impurity scattering length, and W is the sample width.

9.2 Physical Picture

In the hydrodynamic regime:

- Electron-electron (e-e) scattering is very strong.
- Momentum is conserved locally because e-e collisions do not relax total momentum.
- Electrons behave like a viscous fluid where "resistance" arises from internal friction and boundary scattering.

Thus, transport is governed by collective flow rather than independent single-particle motion.

9.3 Boltzmann Equation Perspective

The collision integral separates into momentum-conserving and momentum-relaxing parts:

$$I = I_{ee} + I_{\text{imp}}$$

Electron-electron collisions:

- Conserve total momentum.
- Rapidly establish a moving local equilibrium.

Thus the distribution takes the form:

$$f(\mathbf{r}, \mathbf{k}) = f_0 \left(\varepsilon_k - \mathbf{u}(\mathbf{r}) \cdot \mathbf{k} \right)$$

where $\mathbf{u}(\mathbf{r})$ is the local drift velocity.

9.4 Hydrodynamic Equations

Taking moments of the Boltzmann equation yields fluid equations.

Continuity equation:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0$$

Momentum equation (Navier–Stokes form):

$$mn \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla P + ne\mathbf{E} + \eta \nabla^2 \mathbf{u} - \frac{mn}{\tau_{\text{imp}}} \mathbf{u}$$

where P is pressure and η is the viscosity, which acts as a momentum diffusion coefficient.

9.5 Viscous Flow

In steady state and linear response:

$$\eta \nabla^2 \mathbf{u} - \nabla P + ne\mathbf{E} = 0$$

This is analogous to classical viscous fluid flow.

9.6 Poiseuille Flow

In a channel of width W, the velocity profile becomes parabolic:

$$u(y) \sim y(W - y)$$

leading to:

- Non-uniform current density across the channel.
- Enhanced conductivity compared to the diffusive regime because e-e collisions allow electrons to avoid impurities by flowing collectively.

9.7 Gurzhi Effect

A key signature of hydrodynamic transport is a decrease in resistivity as temperature increases:

$$\rho(T) \sim \frac{1}{\tau_{ee}} \sim T^{-2}$$

This occurs because higher T reduces l_{ee}, making the fluid more "viscous" and better able to flow through the channel without being stopped by impurities, which is opposite to the behavior of normal metals.

9.8 Key Insight

- Transport is a collective fluid phenomenon.
- Momentum-conserving collisions (e-e) dominate the dynamics.
- Viscosity and boundary conditions control the total resistance.

9.9 Experimental Relevance

Hydrodynamic transport has been observed in high-mobility materials:

- Graphene.
- Ultrapure metals (e.g., $PdCoO_2$).
- Two-dimensional electron systems.

10 Limits of Boltzmann Theory

The semiclassical Boltzmann framework breaks down in the following regimes:

- **Quantum oscillations:** Landau quantization (Shubnikov–de Haas) is not captured
- **Localization:** Boltzmann theory assumes electrons are point particles and ignores their wave nature. Hence, phase-coherent interference effects are neglected
- **Strong magnetic fields:** when $\hbar\omega_c \gtrsim k_B T$, semiclassical transport fails.