Intermediate Electromagnetism, UP 203

Problem Set I

1. Prove **Earnshaw's Theorem:** *A charged particle cannot be held in stable equilibrium by electrostatic field alone.* This was demonstrated by the example of eight equal charges placed at the corners of a cube that we discussed in the class. To be more concrete, suppose the eight equal positive charges are placed at $(\pm a/2, \pm a/2, \pm a/2)$. Show that $(0, 0, 0)$ is an equilibrium point. Calculate the potential at $(0, 0, 0)$, $(\epsilon, 0, 0)$, $(\epsilon, \epsilon, 0)$, $(\epsilon, \epsilon, \epsilon)$, where ϵ is small. In which of the three directions $\hat{x}$, $(\hat{x}+\hat{y})$, $(\hat{x}+\hat{y}+\hat{z})$ is the small displacement of a positive test charge stable? (adapted from Problem 3.2 in Griffiths)

2. Prove the following (please explain all the steps logically; Problem 1.1 in Jackson):

   (a) Any excess charge on a conductor must lie on its surface.

   (b) A closed, hollow conductor shields its interior from fields due to external charges, but does not shield its exterior from the fields due to charges places inside it.

   (c) The electric field at the surface of a conductor is normal to the surface and has a magnitude σ/ϵ_0, where σ is the surface charge density per unit area.

3. Consider a general orthogonal coordinate system (u, v, w) with the line element given by $\vec{dl} = U du \hat{u} + V dv \hat{v} + W dw \hat{w}$. Show that

$$\delta^3(\vec{x} - \vec{x}') = \frac{1}{UVW}\delta(u - u')\delta(v - v')\delta(w - w'),$$

   where $\vec{x}' = (u', v', w')$. (adapted from Problem 1.2 in Jackson)

4. Prove the following properties of δ functions:

   (a) $\int f(x)\delta'(x - a)dx = -f'(a)$, where $'$ denotes differentiation w.r.t. argument.

   (b) $x\frac{d}{dx}\delta(x) = -\delta(x)$

   (c) $\frac{d\theta(x)}{dx} = \delta(x)$, where $\theta(x)$ is the step function with $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x \leq 0$.

   (d) $\delta(f(x)) = \sum_i \frac{1}{|\frac{df}{dx}(x_i)|}\delta(x - x_i)$, where $f(x)$ is assumed to have non-vanishing first derivatives at roots x_i.

5. Using δ functions in appropriate coordinates, express the following charge distributions as 3-D charge densities $\rho(\vec{x})$:

   (a) in spherical coordinates, a charge Q uniformly spread over a shell of radius R.

   (b) in cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b.

   (c) in cylindrical coordinates, a charge Q spread uniformly over a flat circular disk of negligible thickness and radius R.

   (d) the same as 5c, but using spherical and cartesian coordinates. (Problem 1.3 in Jackson)

6. The time averaged potential of a neutral hydrogen atom is

$$\phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right),$$

   where q is the magnitude of the electron's charge and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge which results in this potential and interpret your result physically. (Problem 1.5 in Jackson)

7. Two long , cylindrical conductors of radii a_1, a_2 are parallel and separated by a distance $d \gg a_1, a_2$. Show that the capacitance per unit length is given approximately by

$$C \simeq \pi\epsilon_0 \left(\ln \frac{d}{a}\right)^{-1},$$

   where $a = (a_1 a_2)^{1/2}$. (Problem 1.7 in Jackson)

8. Calculate the attractive force between the conductors in the parallel plate capacitor and the parallel cylinder capacitor for (a) fixed charges on each conductor, (b) fixed potential difference between conductors. (Problem 1.9 in Jackson)

9. Prove the *mean value theorem*: For charge-free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of *any* sphere centered on that point. (Problem 1.10 in Jackson)