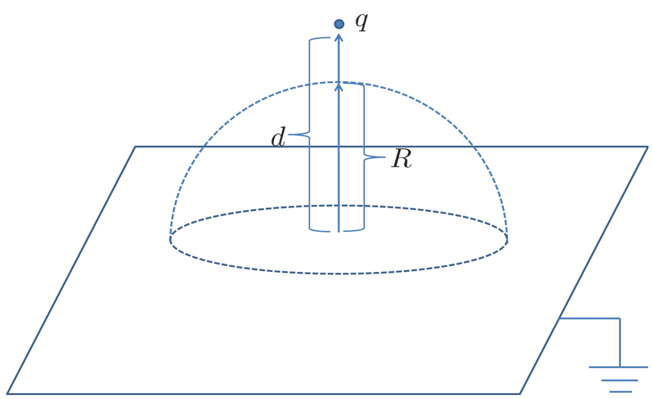
Intermediate Electromagnetism, UP 203

Problem Set II

1. The Green's function for Poisson's equation satisfies $\nabla^2 G(\vec{x}, \vec{x}') = -\delta(\vec{x} - \vec{x}')$. This equation needs to be supplemented by a boundary condition which is satisfied by adding to $G(\vec{x}, \vec{x}')$ a function $F(\vec{x}, \vec{x}')$ that is a solution of the Laplace's equation over the domain. What is the inhomogeneous part of the Green's function (which satisfies the Poisson's equation but not necessarily the boundary conditions) in one, two, three, and n dimensions? For $n \leq 3$ the knowledge of potential due to simple charge distributions will help.

2. The differential equation for the $\Theta(\theta)$ solution in the separation of variables solution for the Laplace's equation is
$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\Theta = 0.$$
Show that the change of variable $x = \cos\theta$ gives
$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + l(l+1)\Theta = 0.$$
Assume a power series solution $\Theta(x) = \sum\limits_{k=0}^{\infty} a_k x^k$ (only non-negative powers appear because $\Theta(x)$ is well-behaved at $x = 0$). Find the series solution by matching coefficients of powers of x. There will be two independent solutions, one with odd powers of x and the other with even powers of x. Show that one of the solutions always diverges at $|x| = 1$ and the other one converges only if l is a non-negative integer and the series terminates at $k = l$. The converging solution is the series expansion of the Legendre polynomial of degree l (containing odd/even powers if l is odd/even).

3. An infinite grounded conducting plane has a hemispherical bump of radius R. A point charge q is placed at a distance d above the centre of the hemisphere as shown in the figure (the line joining the charge and the centre of the hemisphere is perpendicular to the conducting plane).



   (a) Calculate the electric field $\vec{E}(\vec{r})$ everywhere. (problem taken from Subroto Mukerjee)

   (b) What is the total charge on the conducting plane (with the bump)?

4. An insulated , spherical, conducting shell of radius a is in a uniform electric field E_0. If the sphere is cut into two hemispheres by a plane perpendicular to the field, find the force required to prevent the hemispheres from separated (a) if the shell is uncharged and (b) if the total charge on the shell is Q. (Problem 2.9 from Jackson)

5. A hollow cube has conducting walls defined by six planes $x = 0$, $y = 0$, $z = 0$, and $x = a$, $y = a$, $z = a$. The walls at $z = 0$ and $z = a$ are held at a constant potential V. The other four sides are at zero potential. (Problem 2.23 in Jackson)
   (a) Find the potential $\Phi(x, y, z)$ at any point inside the cube.
   (b) Evaluate the potential at the center of cube numerically accurate to three significant figures. How many terms do you need to keep to have this accuracy? Compare this with the potential averaged over the walls. Relate this to the *mean value theorem* in the last HW?
   (c) Find the surface-charge density on the surface $z = a$.

6. Two hemispheres of radius $a$separated at the $z = 0$ plane are maintained at constant potentials $V_0 > 0$ for $z > 0$ and and $-V_0$ for $z < 0$. Find the Green's function (for both $r > a$ and $r < a$) to solve for the potential everywhere in space. Notice that the Green's functions are symmetric in $\vec{x}$ and $\vec{x}'$. Find the potential on the $z-$ axis. Also write series expansion for potential off the $z-$ axis.

7. A capacitor consists of two concentric spheres. Call the inner shell, of radius a, conductor 1, and the outer shell, of radius b, conductor 2. For this two-conductor system, find the coefficients of capacitance C_{11}, C_{22}, and C_{12}. Bonus: Prove that $C_{21} = C_{12}$, in general. This can be proved for a two-conductor system by considering the energy required to raise two conductors to potentials ϕ_{1f} and ϕ_{2f} in two different ways: (a) start from the two conductors at zero potential and raise conductor 1 to ϕ_{1f} keeping conductor 2 at zero potential; then charge conductor 2 to ϕ_{2f} keeping conductor 1 at ϕ_{1f}; (b) carry out the same steps, except that conductor 2 is raised to ϕ_{2f} before conductor 1 is raised to to ϕ_{1f}. (from Purcell)

8. The charge density on the surface of a sphere (radius R) is given by $\sigma = k \cos 3\theta$, where k is a constant. Find the potential inside and outside the sphere, assuming there are no other charges anywhere.

9. The energy density in electrostatic field is non-negative: $w = \epsilon_0 |\vec{E}|^2/2$. The integral of this gives the total energy in electrostatic field, which must be non-negative. The potential energy of an arrangement of two point charges is
$$\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r},$$
which can be negative if the two charges are of opposite sign. How do you reconcile this with the fact that the energy in electrostatic field is non-negative?

10. **Inviscid flow around a spherical sphere:** Start from 2-D Euler equation in steady state and show that the equation for flow pattern ($\vec{u}(r, \theta)$) can be reduced to solving Laplace's equation with boundary conditions. Solve for the velocity distribution.