# Intermediate Electromagnetism, UP 203

## Problem Set VI

Most problems below are adapted from Griffiths.

1. Suppose

$$\vec{E}(r,\theta,\phi,t) = A\frac{\sin\theta}{r}\left[\cos(kr-\omega t) - (1/kr)\sin(kr-\omega t)\right]\hat{\phi}, \text{with } \omega = ck.$$

   This is the simplest spherical wave.

   (a) Show that $\vec{E}$ obeys all Maxwell's equation in vacuum. Find the associated magnetic field.

   (b) Calculate Poynting vector. Average $\vec{S}$ over a full cycle to get the intensity vector $\vec{I}$.

   (c) Calculate the power radiated through a radius $r$. It shouldn't depend on the radius.

2. A steady current flows through a straight wire connected to a capacitor with large plates. Find the electric and magnetic fields in the capacitor as a function of time and distance from the center (assume charge is zero at $t = 0$). Calculate Poynting flux inside the wire (assume it is a cylinder with a finite thickness); where does this energy deposited by Poynting flux go inside the wire? Calculate Poynting flux inside the capacitor (ignore edge effects) and interpret energy conservation. What is the ultimate source of energy?

3. example 8.4 in Griffiths

4. Calculate Maxwell stress tensor ($\overleftrightarrow{T}$) for a simple plane e.m. wave in vacuum. Recall that

$$\frac{\partial}{\partial t}(\mathcal{P}_{\rm mech} + \mathcal{P}_{\rm em}) = \vec{\nabla}\cdot\overleftrightarrow{T}$$

   where $\mathcal{P}_{\rm mech}$ and $\mathcal{P}_{em}$ are mechanical and e.m. momentum densities. Calculate the divergence of Maxwell's tensor for e.m. waves and show that it is consistent with momentum conservation.

5. The dispersion relation of e.m. waves in a metal is given by $k^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$ (recall that $k$ is imaginary in general). Given that the electric field is given by $E_0 e^{-\kappa z}\cos(kz-\omega t)\hat{x}$, find $\kappa$ and $\vec{B}$. For a normally incident e.m. from air to a metal (with conductivity $\sigma$), calculate the energy dissipation rate density ($\sigma|E|^2$) inside the metal. Show that the dissipation rate increases with decreasing conductivity ($\propto 1/\sigma$) and with increasing frequency ($\propto \omega^2$). The implication is that, while waveguides and coaxial cables are good transmitters of low frequency (radio) waves, the dissipation become too much at optical frequencies and its better to switch to an optical fiber.

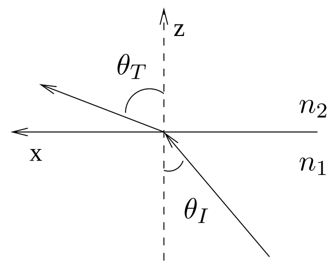
6. (a) Show that the skin depth in poor conductors ($\sigma \ll \omega\epsilon$; whether something is a good conductor or not depends on the frequency!) is $(2/\sigma)\sqrt{\epsilon/\mu}$ (independent of frequency). Calculate it for water ($\mu_r = 1,\ \epsilon_r \approx 80$; $\sigma \approx 4\times 10^{-6}(\Omega m)^{-1}$).

   (b) Show that the skin depth for a good conductor ($\sigma \gg \omega\epsilon$) is $\lambda/2\pi$ ($\lambda$ is the wavelength in the conductor). Find the skin depth for a typical metal ($\sigma \approx 10^7(\Omega m)^{-1}$) in the visible range ($\omega = 10^{15}$/s), assuming $\epsilon_r = \mu_r = 1$. Why are metals opaque?

   (c) Show that in a good conductor the magnetic field lags behind the electric field by $45^0$, and find the ratio of their amplitudes. Show that magnetic field is much larger than electric field for good conductors.

7. Determine the dispersion relation and the mode structure of transverse magnetic (TM) modes ($B_z = 0,\ E_z \neq 0$) in a rectangular waveguide with sides $a$ and $b$. We did the same for TE modes in class. What is the smallest frequency of TM modes that can propagate? What is the ratio of the minimum cut-off frequencies for TM and TE modes?

8. A resonant cavity is formed by closing the open ends of a rectangular waveguide at $z = 0$ and $z = d$. Show that the frequency of resonant modes that represent standing waves inside the resonant cavity is
$$\omega_{lmn} = c\pi\sqrt{(l/d)^2 + (m/a)^2 + (n/b)^2},$$
where $l,\ m,\ n$ are integers. Find the associated TE and TM electric fields. Of course a hollow resonant cavity can't support TEM modes.




9. *Total internal reflection:* Snell's law, $n_1 \sin\theta_I = n_2 \sin\theta_T$, applies to refraction of waves from a medium with refractive index $n_1$ to a medium with index $n_2$; $\theta_I$ and $\theta_T$ are the incident and transmitted angles with respect to the normal. If $n_2 < n_1$, i.e., for propagation into the medium with a lower refractive index, the wave is transmitted at $\theta_T = \pi/2$ for $\sin\theta_{Ic} = n_2/n_1$ (energy does not penetrate into medium 2!). For $\theta_I > \theta_{Ic}$, $\theta_T$ cannot be interpreted as a simple angle. The complex representation of waves $e^{-i(\omega t - k_x x - k_z z)}$ and boundary condition in $x - y$ plane give $k_{xT} = k_{xI} = k_I \cos\theta_I$. Since the frequency in the two media is the same ($\omega$) and the two waves satisfy dispersion relation in their respective media,
$$\omega^2 = c^2 k_I^2/n_1^2 = c^2(k_{xT}^2 + k_{zT}^2)/n_2^2.$$
This gives that the $z-$ wavenumber is complex for $\theta_I > \theta_{I,c}$; $k_{zT} = ik_I(\sin^2\theta_I - (n_2/n_1)^2)$. So the transmitted wave in this case is damping in the $z-$ direction but propagating in the $x-$ direction.

   (a) Fresnel's relation (for electric field lying in the plane) between reflected fields and incident fields is given by (eq. 9.109 in Griffiths) $E_{0R}/E_{0I} = (\alpha - \beta)/(\alpha + \beta)$, where $\beta = \sqrt{\mu_1\epsilon_2/(\mu_2\epsilon_1)}$ and $\alpha = \cos\theta_T/\cos\theta_I$ ($\theta_T$ must be carefully interpreted here!). Show that one gets 100% reflection (this is also true for electric field polarized perpendicular to the plane). How are the phases of reflected and incident electric fields related?

   (b) Find $\vec{E}$ and $\vec{B}$ of the transmitted wave for total internal reflection ($\theta_I > \theta_{Ic}$) and calculate the Poynting flux. Show that on average the evanescent wave carries no energy. Optical fibers are based on the principle of total internal reflection and the losses are minimal (much better than conducting waveguides at optical frequencies).

10. Derive Snell's law for an e.m. wave incident on a conductor with conductivity $\sigma$ at an angle $\theta_I$. Like in total internal reflection, the wave inside the metal is evanescent but it does have a non-zero real part of $k_z$; i.e., transmitted wave carries energy. Show that for a good conductor both the wavelength and the skin depth in the $z-$ direction become quite small.