AA 372

Homework IV

Please submit your codes together with your write-ups. Please email/meet me if something is unclear.

1. **Level Populations, borrowed from a course that I took in Princeton by Prof. Jim Stone:** Solving the level population equations generally requires numerical methods for systems involving more than three levels. Consider a 6-level model of O^{2+} ion. Values for the collisional rate coefficients $\gamma_{i,j}$, Einstein $A-$values $A_{i,j}$, statistical weights g_i, and energy difference between the levels $E_{i,j}$, etc. are given as follows.

   Collision cross-sections at $T = 10^4$ K ($\Omega_{i,j}$): the (i, j) entry in the matrix represents $\Omega_{i,j}$, where $2 \le i \le 6$, $\begin{bmatrix} 0.542 & & & & \\ 0.271 & 1.29 & & & \\ 0.2411 & 0.7233 & 1.2056 & & \\ 0.0307 & 0.0902 & 0.1533 & 0.62 & \\ 0.136 & 0.408 & 0.68 & 0 & 0 \end{bmatrix}$

   Temperature dependence of cross-section ($X_{i,j}$), $\begin{bmatrix} 0 & & & & \\ 0 & 0 & & & \\ 0.14 & 0.14 & 0.14 & & \\ 0.186 & 0.186 & 0.186 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

   Einstein $A-$values $\begin{bmatrix} 2.62 \times 10^{-5} & & & & \\ 3.02 \times 10^{-11} & 9.76 \times 10^{-5} & & & \\ 2.74 \times 10^{-6} & 0.00674 & 0.0196 & & \\ 0 & 0.217 & 0.00052 & 1.78 & \\ 0 & 212 & 522 & 0 & 0 \end{bmatrix}$

   The energy separation of levels in eV, starting from the ground level, are: 0.014 ($E_{1,2}$), 0.0239 ($E_{2,3}$), 2.4751 ($E_{3,4}$), 2.8407 ($E_{4,5}$), 2.11 ($E_{5,6}$). Statistical weights of different levels are $g_1 = 1$, $g_2 = 3$, $g_3 = 5$, $g_4 = 5$ $g_5 = 1$, $g_6 = 1$. All quantities are in CGS units.

   The collision rate coefficient $\gamma_{i,j}$ for downward transitions ($i > j$) is $\gamma_{i,j} = 8.629 \times 10^{-6}\Omega_{i,j}/(g_i T^{1/2})$, while the inverse rate $\gamma_{j,i}$ is related to the forward rate via $\gamma_{j,i} = (g_j/g_i)e^{-E_{i,j}/kT}\gamma_{i,j}$ (the Boltzmann constant $k = 8.62 \times 10^{-5}$ eV/K; notice that the inverse rate is suppressed by the

Boltzmann factor). Use above tables for rate coefficients. Each $\Omega_{i,j}$ has temperature dependence given by $\Omega_{i,j}(T) = \Omega_{i,j}(T/10^4\text{K})^{X_{i,j}}$. Recall that de-excitation happens due to collisions with electrons and spontaneous emission, and excitation happens only due to collisions (ignore photo-absorption and stimulated emission). Thus, in equilibrium rate of population and de-population of every level is equal, or,

$$n_e \sum_{j=1, j\neq i}^{6} n_j \gamma_{j,i} + \sum_{j=i+1}^{6} A_{j,i} n_j = n_i \left(n_e \sum_{j=1, j\neq i}^{6} \gamma_{i,j} + \sum_{j=1}^{i-1} A_{i,j} \right), \text{for } i = 1, .., 6. \tag{1}$$

- These six equations are degenerate because any one equation can be written as a linear combination of the other five. Using the normalization $\sum_{i=1}^{6} n_i = 1$ as a sixth independent equation (n_i represents the fraction of atoms in state i), write a program to solve this system (i.e., any five of these equations and the normalization relation) for the following five pairs of values of n_e and T: $T = 10^4$ and $n_e = 10^3,\ 10^4\ , 10^5$; $n_e = 10^5$ and $T = 5\times10^3,\ 2\times10^4$. For each $n_e,\ T$ pair report the numerical values of n_i, Its best to treat $n_e,\ T$ as input parameters to your code, and simply rerun the code for each value of $n_e,\ T$ pair and record the values of n_i. Please use LAPACK if you are writing a Fortran or C code; you can use the in-built matrix functions in MATLAB if thats your preferred language.
- Ratios of the intensities of the $\lambda5007$ to the $\lambda4363$ lines in the OIII ion (4-3 and 5-4 transitions) are particularly useful for measuring the temperature of the ionized nebula. Plot this ratio versus T over the range 6000 K $< T <$ 20,000 K for $n_e = 100, 10^3$, and 10^4. From this plot, why do you suppose this line ratio is a useful temperature diagnostic? Suppose that electron density in an HII region was found to be $n_e = 10^3$, while $I(\lambda5007)/I(\lambda4363) = 800$ for this same nebula. What is the temperature? Can you suggest a way to measure the electron density (using some other line-ratios)?

2. **Isothermal Parker-Wind:** The simplest (and of course not very realistic) model for the solar wind can be constructed assuming an isothermal (constant temperature) plasma. In steady state ($\partial/\partial t = 0$), the

equations of mass and momentum conservation give,

$$4\pi r^2 \rho v = \dot{M}, \tag{2}$$

$$v\frac{dv}{dr} = -\frac{1}{\rho}\frac{dp}{dr} - \frac{GM_\odot}{r^2}, \tag{3}$$

where $\dot{M}$ is the constant mass outflow rate, v is the radial velocity, r is the radius measured from the center of the sun, p is the gas pressure, and $M_\odot$ is the solar mass. For isothermal equation of state $p = \rho c_s^2$ where $c_s = \sqrt{kT/(\mu m_p)}$ is the constant sound speed. The momentum equation can be written as

$$(v^2 - c_s^2)\frac{d\ln v}{dr} = 2\frac{c_s^2}{r} - \frac{GM_\odot}{r^2}. \tag{4}$$

The sonic point r_s is the radius where $v = c_s$, and both LHS and RHS vanish. Thus, $r_s = GM/2c_s^2$. The momentum equation can be written in the Bernoulli form (generalized energy conservation) as

$$\mathrm{Be} \equiv \frac{v^2}{2} - 2c_s^2 \ln r - c_s^2 \ln v - \frac{GM_\odot}{r} = \frac{c_s^2}{2} - 2c_s^2 \ln r_s - c_s^2 \ln c_s - \frac{GM_\odot}{r_s}. \tag{5}$$

The above equation is an implicit equation for velocity v as a function of radius for the isothermal wind. You will need to find the appropriate root of Eq. 5. Recall that the wind speed goes from subsonic to supersonic at the sonic point. The solution is solely determined by the temperature. Solve for the wind velocity v for three temperatures $T = 10^5$, 10^6, & 10^7 K. Find the solutions from $r = R_\odot$ to $r = 200R_\odot$. Plot the solar wind speed and density (assuming that $\dot{M} = 10^{-14}M_\odot/\mathrm{yr}$, and using Eq. 2) as a function of radius. Choose a grid which is logarithmically spaced in radius; i.e., $d\ln r$ is constant. Use solar radius $R_\odot = 7\times10^{10}$ cm, mean mass per particle $\mu = 0.5$, proton mass $m_p = 1.67 \times 10^{-24}$ g, solar mass $M_\odot = 2 \times 10^{33}$ g, Boltzmann constant $k = 1.4 \times 10^{-16}$ erg K^{-1}.

3. Using Taylor series expansions (or otherwise) prove the following for different root-finding algorithms

   (a) The number of steps to converge to a given tolerance in the root (relative error ϵ) is roughly $-\log_2 \epsilon$.

(b) The errors in subsequent steps scale as $\epsilon_{k+1} = \epsilon_k^1.618$ for Secant method.

(c) $\epsilon_{i+1} = -\epsilon_i^2 f''(x)/2f'(x)$ for Newton-Raphson method.