AA 372

Homework V

Please submit your codes together with your write-ups. Please email/meet me if something is unclear. **This problem set is adapted from Prof. Jim Stone's course at Princeton.**

1. **Planetary Orbits:** Solve for the motion of a point mass in a $1/r^2$ gravitational force. Of course this is the classic two-body problem and we know analytic solutions. Orbits are ellipses if the total energy is negative. Treat this as an initial value problem and compare different ODE solvers to the analytic solution.

   (a) Derive the equations of motion in the $x - y$ plane perpendicular to the angular momentum vector. Choose your coordinates such that the orbit is confined to this plane. Write four first order ODEs for dx/dt, dy/dt, dv_x/dt, and dv_y/dt. For convenience you can choose G, M, etc. parameters to be unity.

   (b) Now integrate for 25 orbits using initial conditions corresponding to a circular orbit with different methods: forward Euler, backward Euler, leap-frog, and RK4. Plot orbits on $x - y$ plane and compare results. Choose a constant timestep, say a tenth of the orbital time. Plot error (e.g., rms error in particle position, using the analytic solution) in each method as a function of time. Which method gives highest accuracy for a given CPU time? Do any of the methods conserve energy, angular momentum (plot these conserved quantities as a function of time for different methods)? Try the leap-frog method with a varying timestep (say $dt = T[1+0.1*rand]/10$, where $rand$ is a uniform random number between 0 and 1) and see how the energy conservation looks like. Basic leap-frog maintains time reversibility (and other attractive properties) only for evolution using equal timesteps.

   (c) Repeat (b) for initial conditions corresponding to an elliptical orbit with eccentricity, $e = \sqrt{1 - b^2/a^2} = 0.9$, where a, b are semi-major and semi-minor axes of the ellipse. Are errors bigger/smaller for an elliptical orbit?

   Please write your own forward/backward Euler and leap-frog routines. It is fine to use black-box (e.g., adapted from NR) RK4 routines.

2. **Lane-Emden equations:** The equations of stellar structure can be simplified enormously if we assume a polytropic equation of state ($p = K\rho^{(1+1/n)}$), where $n = 1/(\gamma - 1)$ is related to the polytropic index. The equations for mass conservation and hydrostatic equilibrium are:

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho, \tag{1}$$

$$\frac{dp}{dr} = -\frac{GM(r)}{r^2}\rho \tag{2}$$

We can define new variables θ, ξ such that, $\rho = \lambda\theta^n$, $p = K\lambda^\gamma\theta^{n+1}$, and $r = \xi\sqrt{\frac{(1+n)K}{4\pi G}\lambda^{(1/n-1)}}$. The scaled mass is defined as $M = m4\pi\lambda\left[\frac{(1+n)K}{4\pi G}\lambda^{(1/n-1)}\right]^{3/2}$. Thus, the normalized first order Lane-Emden equations are:

$$\frac{dm}{d\xi} = \xi^2\theta^n, \tag{3}$$

$$\frac{d\theta}{d\xi} = -\frac{m}{\xi^2}., \tag{4}$$

subject to boundary conditions $m = 0$ at $\xi = 0$ (mass enclosed vanishes at the origin), and $\theta = 0$ at $\xi = \xi_\star$ (density vanishes at the stellar surface). Solve the Lane-Emden equations for index $n = 3$. The difficulty with solving this equation is that $\xi_\star$ is not know a priori.

(a) First, try solving the system using 'shooting by hand.' Integrate Eqs. 3 & 4 as an initial value problem using θ at $\xi = 0$ between 1 and 10. In each case plot the solution from $\xi = 0$ to $\xi_\star$ ($\xi_\star$ will be different for each initial guess; a bigger star for a larger core temperature) on the same graph. Estimate roughly the value of θ at $\xi = 0$ that gives $\xi_\star = 2$. Because of coordinate singularity at $\xi = 0$, you will need to start your integrations from $r = \epsilon$ where $\epsilon \ll 1$. Taylor series expansion for m, θ, using Eqs. 3 & 4, gives $m \approx m(0) + (dm/d\xi)_0\epsilon + ... = \epsilon^3\theta_0^3$ and similarly $\theta \approx \theta_0$, where $\theta_0 = \theta(0)$.

(b) Now implement an automated shooting method using Newton-Raphson iteration that finds the value of θ at $\xi = 0$ which gives $\xi_\star = 2$. How does your value differ from the intuitive approach in (a)?