AA 372

Homework VI

Please submit your codes together with your write-ups. Please email/meet me if something is unclear.

1. **Different methods for hyperbolic equations:** We have discussed several methods for solving hyperbolic equations. We will apply some of the methods that we discussed to two prototype hyperbolic equations: the advection equation and the Burger's equation.

   The constant velocity advection equation is

   $$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0, \tag{1}$$

   where v is the constant advection velocity. Assume $v = 1$, periodic boundary conditions, domain $0 \leq x \leq 1$, and assume that the initial condition is

   $$f(x, t = 0) = e^{-200(x-0.3)^2} + \mathrm{sqr}[0.6, 0.8], \tag{2}$$

   where $\mathrm{sqr}[a, b]$ is a square wave for $a \leq x \leq b$; i.e., it is zero everywhere except when x lies between a and b, where it is 1. Plot $f(x, t)$ at $t = 0,\ 1,\ 2,\ 3$. Compare the results with the following methods (use $\Delta t = 0.5\Delta x/v$; use 128 grid points): upwind, Lax, and Lax-Wendroff. Which method is the best and why? Try combining Lax-Wendroff with artificial viscosity (discussed later) and see if the oscillations go away. Perform convergence analysis for all these methods at $t = 1$ (use 32, 64, 128, 256, 512 grid points) and plot L1 error (you can either use the analytic results or use Richardson error; i.e., treating the higher resolution result as the true solution) as a function of resolution (Δx). What's the order of convergence?

   The Burger's equation, a model for supersonic flows, is given by

   $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{3}$$

   where u is the fluid velocity. Its better to numerically evolve the conservative form,

   $$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0. \tag{4}$$

Use periodic boundary conditions, $0 \leq x \leq 1$, and with the initial condition given by

$$u(x, t = 0) = \sin(2\pi x). \tag{5}$$

Show results at $t = 0,\ 0.1,\ 0.25,\ 0.5$ using Lax, upwind, Lax-Wendroff methods; use $\Delta t = 0.5\Delta x$ and resolution of 128 grid-points. You will see that the Lax-Wendroff method becomes unstable once a shock forms. Lax-Wendroff method can be stabilized by using artificial viscosity which is implemented as follows. Instead of solving Eq. 4, we solve

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = -\frac{\partial q}{\partial x}, \tag{6}$$

where q is the (artificial) viscous flux given by

$$\begin{aligned} q &= l^2 \left(\frac{\partial u}{\partial x}\right)^2 \text{ when } \frac{\partial u}{\partial x} < 0, \\ &\quad 0, \text{ otherwise,} \end{aligned} \tag{7}$$

where l is a length scale over which a shock is smoothened. While both the transport and diffusion terms in Eq. 6 can be applied in a single step with both terms treated explicitly (forward in time), remember that the diffusion term has a restrictive stability limit on Δt. Thus it is better to use operator splitting and applying the transport and viscous diffusion steps independently. Since the main timestep ($\Delta t = 0.5\Delta x$) can be much shorter than the viscous timestep ($\Delta t_{\rm vis} = 0.5\Delta x^2/(l^2 \max[\partial u/\partial x]) \approx 0.25\Delta x/(\text{qcon } \max[u])$, where qcon$=l/\Delta x$), we have to subcycle the artificial viscosity step; i.e., apply artificial viscosity for multiple (nsub) times with a smaller timestep ($\Delta t_{\rm sub}$) such that $\Delta t_{\rm sub} \leq \Delta t_{\rm vis}$ and nsub $\times\ \Delta t_{\rm sub} = 0.5\Delta x$ (the main timestep). What value of qcon is able to make the scheme stable? Plot the results with this method at different times. Which scheme seems to work best?

2. **Solving Poisson equation:** The gravitational potential ($\Phi(x, y)$) due to a mass density distribution $\rho(x, y)$ is the solution of the following Poisson equation,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \rho \tag{8}$$

in appropriate units. The finite difference form for this equation in two dimensions is

$$\frac{\Phi_{i+1,j} - 2\Phi_{i,j} + \Phi_{i-1,j}}{\Delta x^2} + \frac{\Phi_{i,j+1} - 2\Phi_{i,j} + \Phi_{i,j-1}}{\Delta y^2} = \rho_{i,j}. \tag{9}$$

The matrix equation corresponding to this equation is not easy to solve directly (its easy in 1D because we have a tridiagonal matrix), so its solved iteratively. Recall Jacobi relaxation which solves the following equation iteratively until the desired accuracy is obtained

$$\Phi_{i,j}^{n+1} = \frac{(\Phi_{i+1,j}^{n} + \Phi_{i-1,j}^{n} + \Phi_{i,j+1}^{n} + \Phi_{i,j-1}^{n})}{4} - \frac{\Delta^2 \rho_{i,j}}{4}, \tag{10}$$

where $\Delta = \Delta x = \Delta y$. Solve the above equation using periodic boundary conditions; the density distribution is given by

$$\begin{aligned} \rho(x,y) &= e^{-y/0.1} \text{ if } -0.3 \leq x \leq 0.3, \\ &\quad 0 \text{ otherwise.} \end{aligned} \tag{11}$$

Iterate Eq. 10 and make contourplots of the potential (Φ) at 10^{th}, 100^{th} and 1000^{th} iteration on a 40×40 grid. Plot L1 error (using consecutive iterations) as a function of number of iterations. From these contourplots can you see from that the large k (small scale) modes converge quickly but the large scale modes require many iterations to converge. This can be seen by realizing that Jacobi method is equivalent to solving the diffusion equation explicitly with a stable timestep; thus diffusion at large scales takes many timesteps to capture as compared to small scales.

Try to solve the same problem with Gauss-Seidel relaxation (its only very slightly different from Jacobi in that the Φ values are updated right away); it corresponds to the matrix decomposition (see class-slides and NR for details)

$$(\mathbf{L} + \mathbf{D}) \cdot \mathbf{x}^{(r)} = -\mathbf{U} \cdot \mathbf{x}^{(r-1)} + \mathbf{b}, \tag{12}$$

where matrix $\mathbf{A}$ is written as the sum of upper/lower triangular and diagonal matrices. Successive over-relaxation (SOR) method is equivalent to

$$\mathbf{x}^{(r+1)} = \mathbf{x}^{(r)} + \omega(\mathbf{L} + \mathbf{D})^{-1} \cdot \mathbf{r}^{(r-1)}, \tag{13}$$

where $\mathbf{r} \equiv \mathbf{b} - \mathbf{A} \cdot \mathbf{x}$ is the residual, and ω is the over-relaxation parameter; $(\mathbf{L}+\mathbf{D})^{-1} \cdot \mathbf{r}^{(r-1)}$ can be easily evaluated via forward-substitution because $(\mathbf{L}+\mathbf{D})$ is a very simple lower-triangular matrix. The optimum choice of ω for Poisson equation in Cartesian coordinates is $\omega \approx 2/(1+\pi/J)$ where J is the number of grid points in each direction. Apply SOR for the above problem and plot L1 error as a function of number of iterations. Does SOR converge faster than Jacobi, as advertized? Try $\omega = 0.5, 1.5$ and see how these compare to the optimum case.