FERMIONS IN DISORDER

Vijay B. Shenoy
(shenoy@physics.iisc.ernet.in)

Centre for Condensed Matter Theory
Department of Physics
Indian Institute of Science

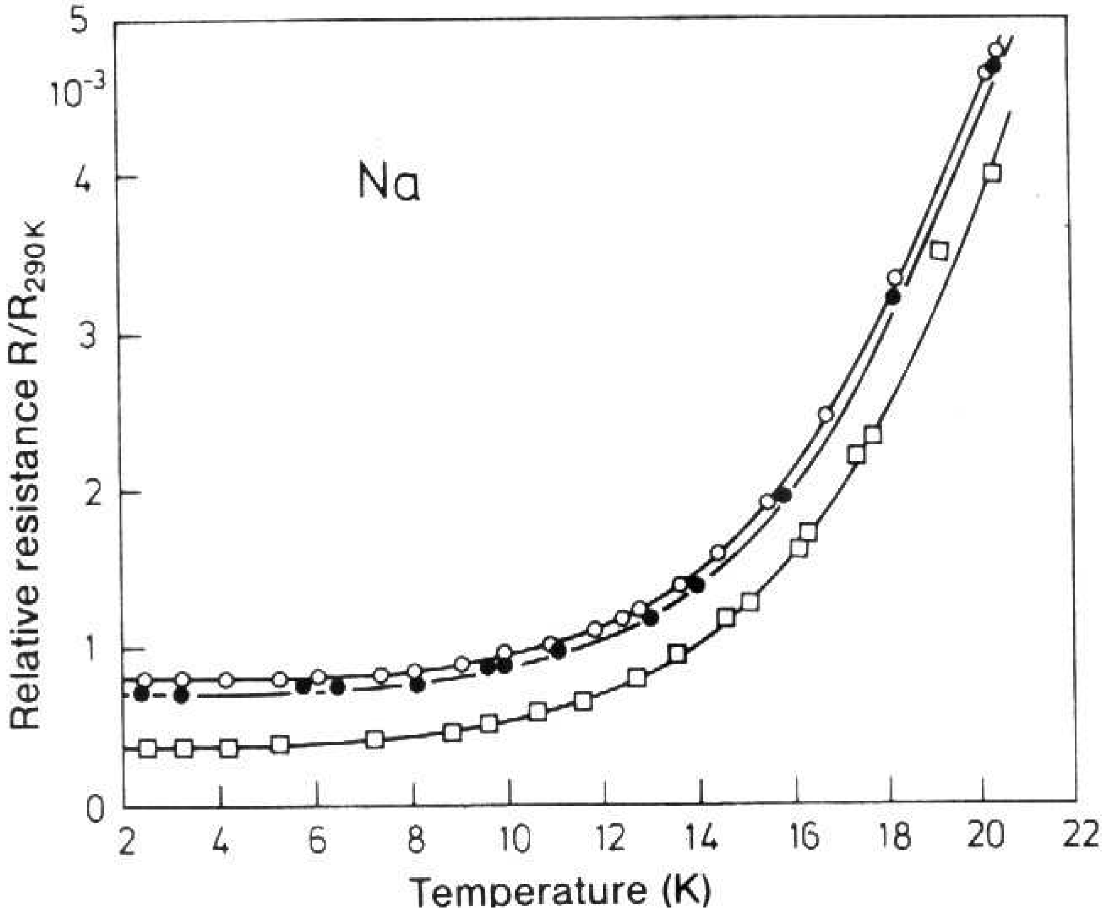
Overview

- Recap: Linear Response Theory et al.
- What do we mean by "disorder"? Why bother?
- Synopsis of The Story
- A More Detailed Plan
    - Preliminary Physical Discussion: Hamiltonians, Localized States etc.
    - "Gang of Four" Scaling Theory
    - Self Consistent Theory of Localization
    - Replica Trick
    - Field Theory of Disorder Problem

Metals, Wonder Materials!

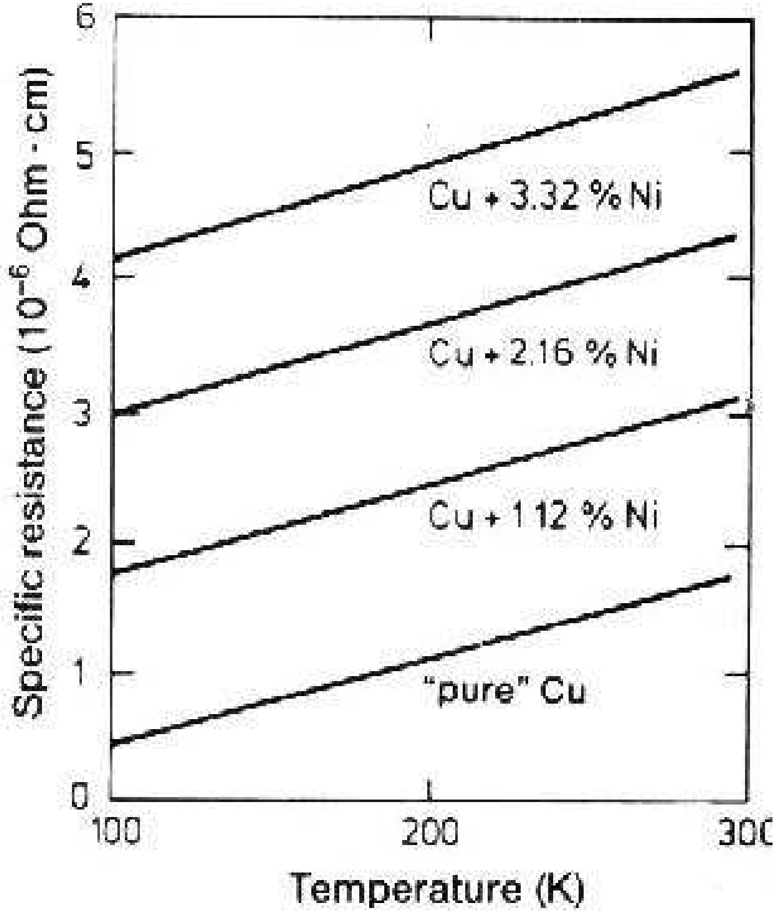
- **What is a metal?**
- **Many electron system with a Fermi Surface**
- **Key:** *Gapless excitations*
  - **Linear T specific heat**
  - **Temperature independent magnetic susceptibility**
  - **...**
  - **"Protected" from repulsive interactions by Pauli – Fermi liquid**

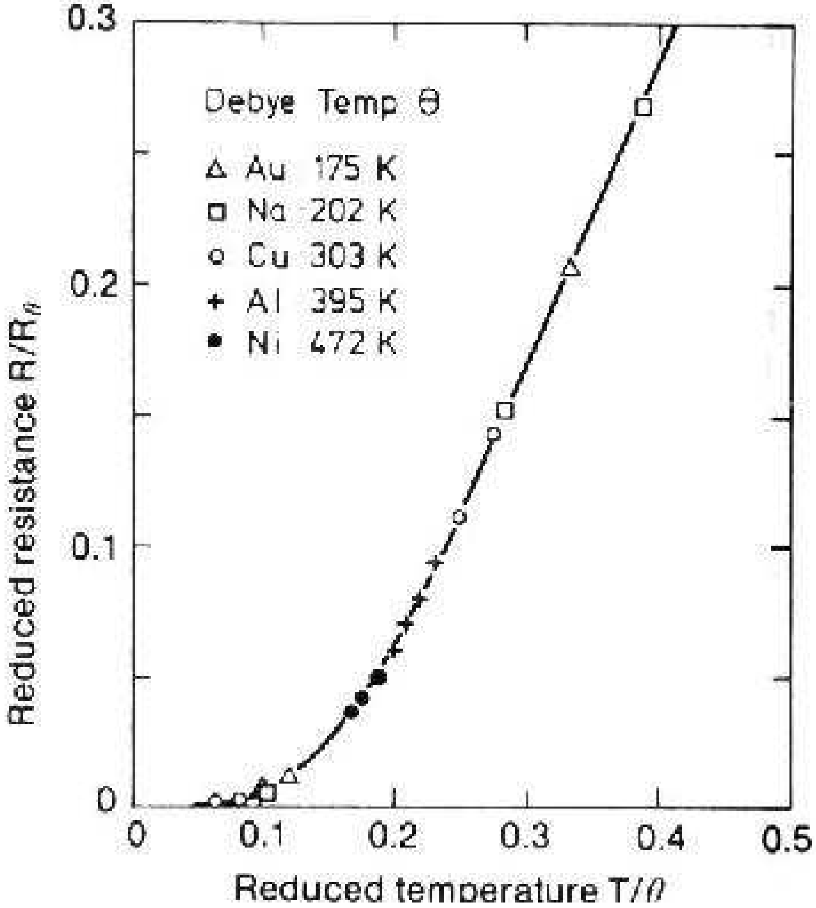
Resistivity in Metals



- **Almost constant at “low” temperatures...all way to linear at high temperatures**

Resistivity in Metals...There's More!





- **Increases with impurity content**
- **Has some "universal" features...**

Our Immediate Questions...and Answers

- What *is* resistivity anyway? – Linear response..
- “Simplest” ideas about resistivity
    - Drudé theory
    - Bloch-Boltzmann theory
- Crux: ”Semi”classical ideas, mean free path, relaxation time...

Linear Response

- **Stimulus (E-field) may vary in space and time** $\boldsymbol{E}(\boldsymbol{r}, t)$
- **Response (current) also varies in space and time** $\boldsymbol{j}(\boldsymbol{r}, t)$
- **What is the most general** *linear* **response?**
- **The most general linear response is** *non-local* **in** *both* **space and time**

$$\boldsymbol{j}(\boldsymbol{r}, t) = \int \mathrm{d}^3\boldsymbol{r}' \int \mathrm{d}t' \, \boldsymbol{\sigma}(\boldsymbol{r}, t|\boldsymbol{r}', t')\boldsymbol{E}(\boldsymbol{r}', t')$$

- **The conductivity tensor response function** $\boldsymbol{\sigma}(\boldsymbol{r}, t|\boldsymbol{r}', t')$ **is a** *property of our system (material)* **– notice the** *nonlocality* **of response**
- **In “nice” systems (“time-invariant and translationally invariant”)** $\boldsymbol{\sigma}(\boldsymbol{r}, t|\boldsymbol{r}', t') = \boldsymbol{\sigma}(\boldsymbol{r} - \boldsymbol{r}', t - t')$

What are we measuring in experiments?

- **The conductivity tensor response function in nice systems can be written in Fourier space $\boldsymbol{\sigma}(\boldsymbol{q},\omega)$**
- **The complex amplitude $\boldsymbol{j}(\boldsymbol{q},\omega)$ of the current response for an electric field $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{q},\omega)e^{i(\boldsymbol{q}\cdot\boldsymbol{r}-\omega t)}$ is given as $\boldsymbol{j}(\boldsymbol{q},\omega) = \boldsymbol{\sigma}(\boldsymbol{q},\omega)\cdot\boldsymbol{E}(\boldsymbol{q},\omega)$**
- **Imagine $\boldsymbol{q} \longrightarrow 0, \omega \longrightarrow 0$, i.e., a "constant" electric field; the response is described be the complex tensor $\boldsymbol{\sigma}(\boldsymbol{q} \rightarrow 0, \omega \rightarrow 0)$ (note: order of limit is crucial, more later)**
- **Assume isotropic system, then σ (conductivity) is simply $\Re\sigma(\boldsymbol{q} \rightarrow \boldsymbol{0}, \omega \rightarrow 0)$**
- **Resistivity $\rho = 1/\sigma$!**

Drudé Theory – Review

- **Electrons: a classical gas**
- **Collision time τ, gives the equation of motion**

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{\boldsymbol{p}}{\tau} + \boldsymbol{F}$$

  $\boldsymbol{p}$ – momentum, $\boldsymbol{F}$ – “external” force

- **Gives the “standard result” for conductivity**

$$\sigma = \frac{ne^2\tau}{m}$$

  (all symbols have usual meanings)

- **All is, however, *not* well with Drudé theory!**

Bloch Theory

- **We do need quantum mechanics to understand metals (all materials, in fact)**
- **In the periodic potential of the ions, wave functions are $\psi_{\boldsymbol{k}}(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} u_{\boldsymbol{k}}(\boldsymbol{r})$ ($u_{\boldsymbol{k}}$ is a lattice periodic function), $\boldsymbol{k}$ is a vector in the 1st Brillouin zone**
- **The Hamiltonian expressed in Bloch language $H = \sum_{\boldsymbol{k}\sigma} \varepsilon(\boldsymbol{k}) |\boldsymbol{k}\rangle\langle\boldsymbol{k}|$ (one band), $\varepsilon(\boldsymbol{k})$ is the band dispersion (set aside spin throughout these lectures!)**
- **"Average velocity" in a Bloch state $\boldsymbol{v}(\boldsymbol{k}) = \dfrac{1}{\hbar}\dfrac{\partial\varepsilon}{\partial\boldsymbol{k}}$**
- **Occupancy of a Bloch state $f^0(\boldsymbol{k}) = \dfrac{1}{e^{\beta(\varepsilon(\boldsymbol{k})-\mu)} + 1}$, $\beta = 1/(k_B T)$, μ – chemical potential**

So, what is a metal?

- **Chemical potential μ determined from electron concentration**
- **Try to construct a surface in the reciprocal space such that $\varepsilon(\boldsymbol{k}) = \mu$**
- **If such a surface exists (at $T = 0$) we say that the material is a** *metal*
- **A metal has a** *Fermi surface*
- **Ok, so how do we calculate conductivity?**
- **Need to understand "how electron moves" under the action of "external forces"**

Semi-classical Electron Dynamics

- **Key idea: External forces (F; electric/magnetic fields) cause transition of electronic states**
- **Rate of transitions $\hbar \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = \boldsymbol{F}$ – Quantum version of "Newton's law"**
- **By simple algebra, we see the "acceleration" $\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{M}^{-1}\boldsymbol{F}$, $\boldsymbol{M}^{-1} = \frac{1}{\hbar^2}\frac{\partial^2 \varepsilon}{\partial \boldsymbol{k} \partial \boldsymbol{k}}$**
- **Electron becomes a "new particle" in a periodic potential! Properties determined by value of M at the chemical potential**
- **But, what about conductivity? If you think about this, you will find a very surprising result! (Essentially infinite!)**

Conductivity in Metals

- What makes for *finite* conductivity in metals?
- Answer: "Collisions"
- Electrons may scatter from impurities/defects, electron-electron interactions, electron-phonon interaction etc...
- How do we model this? Brute force approach of solving the full Schrödinger equation is highly impractical!
- Key idea: The electron gets a "life-time" – i.e., an electron placed in a Bloch state $\boldsymbol{k}$ evolves according to $\psi(t) \sim \psi_{\boldsymbol{k}} e^{-i\varepsilon(\boldsymbol{k})t - \frac{t}{2\tau_{\boldsymbol{k}}}}$; "lifetime" is $\tau_{\boldsymbol{k}}$!
- Conductivity could plausibly be related to $\tau_{\boldsymbol{k}}$; how?

Boltzmann Theory

- **Nonequilibrium distribution function $f(\boldsymbol{r}, \boldsymbol{k}, t)$:**
  - **"Occupancy" of state $\boldsymbol{k}$ at position $\boldsymbol{r}$ and time t**
  - **$\boldsymbol{r}$ in $f(\boldsymbol{r}, \boldsymbol{k}, t)$ represents a suitable "coarse grained" length scale (much greater than the atomic scale) such that "each" $\boldsymbol{r}$ represents a thermodynamic system**
- **Idea 1: The (possibly nonequilibrium) state of a system is described by a** *distribution function* $f(\boldsymbol{r}, \boldsymbol{k}, t)$
- **Idea 2: In equilibrium, $f(\boldsymbol{r}, \boldsymbol{k}, t) = f^0(\boldsymbol{k})$! External forces act to drive the distribution function** *away* **from equilibrium!**
- **Idea 3: Collisions act to** *"restore"* **equilibrium – try to bring f back to f^0**

Time Evolution of $f(\boldsymbol{r}, \boldsymbol{k}, t)$

- **Suppose we know f at time $t = 0$, what will it be at a later time t if we know all the "forces" acting on the system?**
- **Use semi-classical dynamics: An electron at $\boldsymbol{r}$ in state $\boldsymbol{k}$ at time t was at $\boldsymbol{r} - \boldsymbol{v}\Delta t$ in the state $\boldsymbol{k} - \frac{\boldsymbol{F}}{\hbar}\Delta t$ at time $t - \Delta t$**
- **Thus, we get the Boltzmann transport equation**

$$f(\boldsymbol{r}, \boldsymbol{k}, t) = f(\boldsymbol{r} - \boldsymbol{v}\Delta t, \boldsymbol{k} - \frac{\boldsymbol{F}}{\hbar}\Delta t, t - \Delta t) \quad + \quad \left.\frac{\partial f}{\partial t}\right|_{coll.} \Delta t$$

$$\Longrightarrow \frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \frac{\boldsymbol{F}}{\hbar} \cdot \frac{\partial f}{\partial \boldsymbol{k}} \quad = \quad \left.\frac{\partial f}{\partial t}\right|_{coll.}$$

- **If we specify the forces and the collision term, we have an initial value problem to determine $f(\boldsymbol{r}, \boldsymbol{k}, t)$**

Electrical Conductivity

- **BTE becomes**

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \frac{\boldsymbol{F}}{\hbar} \cdot \frac{\partial f}{\partial \boldsymbol{k}} = -\frac{f - f^0}{\tau_{\boldsymbol{k}}}$$

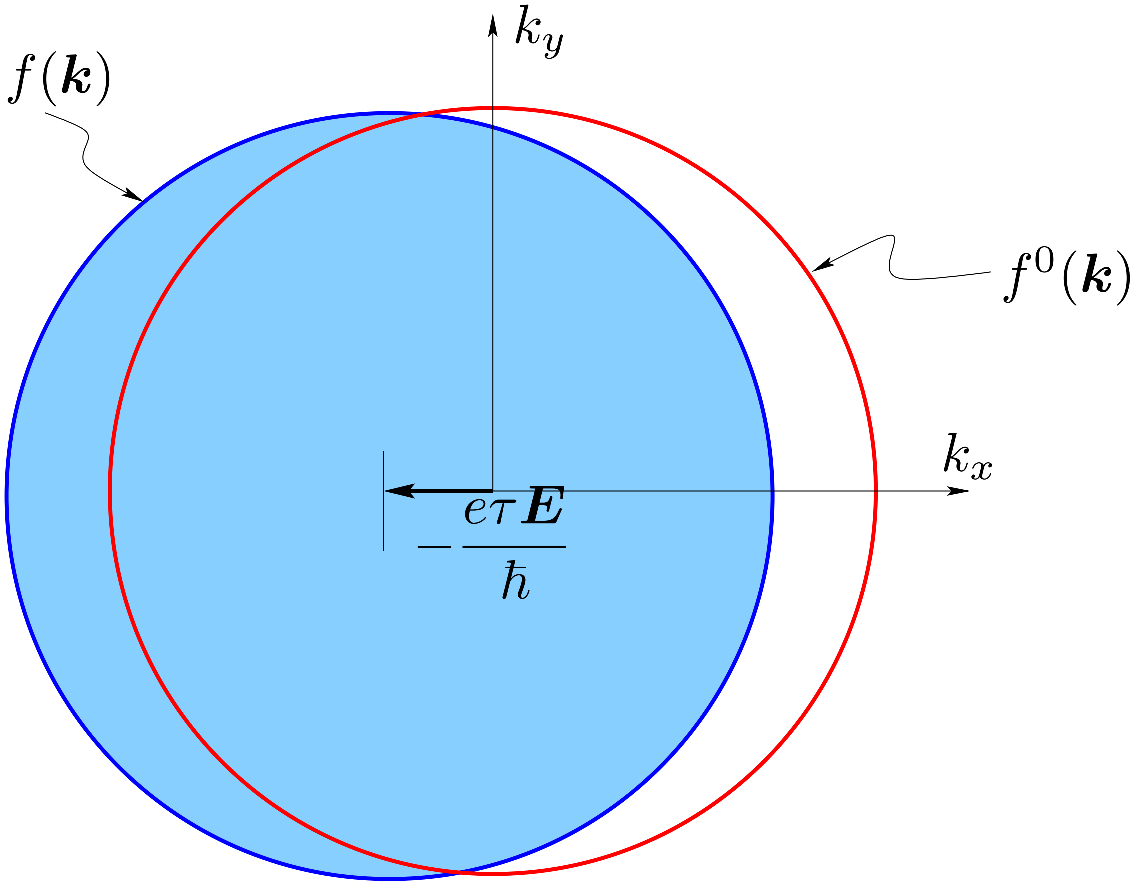
- **Homogeneous DC electric field** $\boldsymbol{F} = -e\boldsymbol{E}$
- **We look for the** *steady homogeneous* **response**

$$\frac{\boldsymbol{F}}{\hbar} \cdot \frac{\partial f}{\partial \boldsymbol{k}} = -\frac{f - f^0}{\tau_{\boldsymbol{k}}} \implies f = f^0 - \frac{\tau_{\boldsymbol{k}} \boldsymbol{F}}{\hbar} \cdot \frac{\partial f}{\partial \boldsymbol{k}}$$

- **Approximate solution (Exercise: Work this out)**

$$f(\boldsymbol{k}) \approx f^0 + \frac{e\tau_{\boldsymbol{k}} \boldsymbol{E}}{\hbar} \cdot \frac{\partial f^0}{\partial \boldsymbol{k}} \approx f^0 \left(\boldsymbol{k} + \frac{e\tau_{\boldsymbol{k}} \boldsymbol{E}}{\hbar} \right)$$

Solution of BTE



- **Fermi surface "shifts" (Exercise: *estimate* order of magnitude of shift)**

Conductivity from BTE

- **Current**

$$\boldsymbol{j} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\boldsymbol{k}\, (-e\boldsymbol{v})\, \frac{e\tau_{\boldsymbol{k}}\boldsymbol{E}}{\hbar} \cdot \frac{\partial f^0}{\partial \boldsymbol{k}}$$

- **Conductivity tensor**

$$\boldsymbol{\sigma} = -\frac{1}{(2\pi)^3}\frac{e^2}{\hbar} \int \mathrm{d}^3\boldsymbol{k}\ \tau_{\boldsymbol{k}}\ \boldsymbol{v}\ \frac{\partial f^0}{\partial \boldsymbol{k}}$$

- **Further, with spherical Fermi-surface (free electron like), $\tau_{\boldsymbol{k}}$ roughly independent of $\boldsymbol{k}$ (Exercise: Show this)**

$$\boldsymbol{\sigma} = \frac{ne^2\tau}{m}\mathbb{1}$$

This looks strikingly close to the Drudé result, but the physics could not be more different!

What about experiments?

- **Well, we now have an expression for conductivity; we should compare with experiments?**
- **What determines the T dependence of conductivity? Yes, it** *is essentially* **the T dependence of τ (***only in metals***)**
- **But we do not yet have τ!!**
- **Need a way to calculate τ...**
- **...**
- **Revisit the idea of electron-lifetime...how do we calculate life time of an electron?**

Lifetime due to Impurity Scattering

- **Impurity potential V_I, causes transitions from one Bloch state to another**
- **Rate of transition from $\boldsymbol{k} \to \boldsymbol{k}'$**

$$W_{\boldsymbol{k}\to\boldsymbol{k}'} = \frac{2\pi}{\hbar} |\langle \boldsymbol{k}' | V_I | \boldsymbol{k} \rangle|^2 \delta(\varepsilon(\boldsymbol{k}') - \varepsilon(\boldsymbol{k}))$$

- **Total rate of transition, or inverse lifetime**

$$\frac{1}{\tau^I_{\boldsymbol{k}}} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\boldsymbol{k}' \, W_{\boldsymbol{k}\to\boldsymbol{k}'}$$

- **Can we use $\tau^I_{\boldsymbol{k}}$ as the τ in the Boltzmann equation?**
- **Ok in order of magnitude, but not alright! Why?**

How to calculate τ?

- **Look back at the collision term, can write it more elaborately as**

$$\left.\frac{\partial f}{\partial t}\right|_{coll.} = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\boldsymbol{k}'\, W_{\boldsymbol{k}\to\boldsymbol{k}'}\,\bigl(f(\boldsymbol{k})(1-f(\boldsymbol{k}')) - f(\boldsymbol{k}')(1-f(\boldsymbol{k}$$

$$= \frac{1}{(2\pi)^3}\int \mathrm{d}^3\boldsymbol{k}'\, W_{\boldsymbol{k}\to\boldsymbol{k}'}\,\bigl(f(\boldsymbol{k}) - f(\boldsymbol{k}')\bigr)$$

Note that k and k' are of the same energy

- **Take $\tau_{\boldsymbol{k}}$ to depend only on $\varepsilon(\boldsymbol{k})$**
- **Now,** $(f(\boldsymbol{k}) - f(\boldsymbol{k}')) \approx -\frac{\tau e}{\hbar}\frac{\partial f^0}{\partial \varepsilon}\bigl(\boldsymbol{v}(\boldsymbol{k}) - \boldsymbol{v}(\boldsymbol{k}')\bigr)\cdot \boldsymbol{E}$

Calculation of τ cont'd

- **Putting it all together**

$$\begin{aligned}
-\frac{e}{\hbar}\frac{\partial f^0}{\partial \varepsilon}\boldsymbol{v}(\boldsymbol{k})\cdot\boldsymbol{E} &= -\frac{1}{(2\pi)^3}\frac{\tau e}{\hbar}\frac{\partial f^0}{\partial \varepsilon}\int \mathrm{d}^3\boldsymbol{k}'\, W_{\boldsymbol{k}\to\boldsymbol{k}'}\left(\boldsymbol{v}(\boldsymbol{k})-\boldsymbol{v}(\boldsymbol{k}')\right)\cdot \\
\Longrightarrow \frac{1}{\tau} &= \frac{1}{(2\pi)^3}\int \mathrm{d}^3\boldsymbol{k}'\, W_{\boldsymbol{k}\to\boldsymbol{k}'}\left(1-\frac{\boldsymbol{v}(\boldsymbol{k}')\cdot\hat{\boldsymbol{E}}}{\boldsymbol{v}(\boldsymbol{k})\cdot\hat{\boldsymbol{E}}}\right) \\
\Longrightarrow \frac{1}{\tau} &= \frac{1}{(2\pi)^3}\int \mathrm{d}^3\boldsymbol{k}'\, W_{\boldsymbol{k}\to\boldsymbol{k}'}\left(1-\cos\left(\widehat{\boldsymbol{k},\boldsymbol{k}'}\right)\right)
\end{aligned}$$

- **Note τ is *different* from the "quasiparticle" life time!**
- **Key physical idea: Forward scattering *does not* affect electrical conductivity!**

T dependence of τ

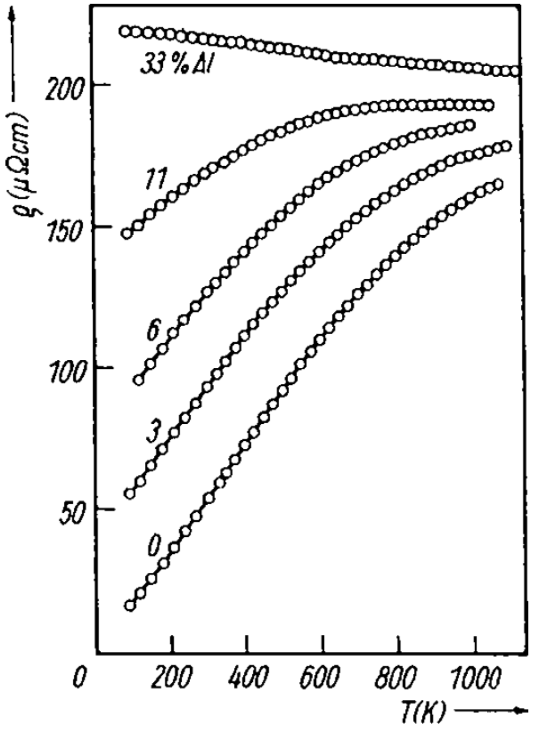
- T dependence strongly depends on the mechanism of scattering
- Common scattering mechanisms
  - Impurity scattering
  - e–e scattering $\sim T^2$
  - e–phonon scatting ($\sim T^5$ low T, $\sim T$ high T)
- More than one scattering mechanism may be operative; one has an effective τ (given by the Matthiesen's rule)

$$\frac{1}{\tau} = \sum_i \frac{1}{\tau_i}$$

- Explains universal behaviour of good metals! So what remains?

Experiments and Puzzles

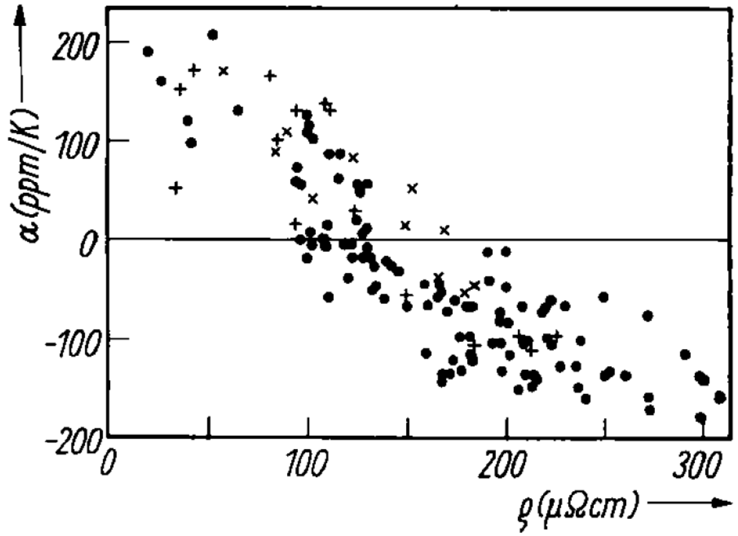
- **Numbers: Typical metals - $\rho \sim 10^{-6(-8)}$ Ohm-cm(m)**
- **Data by Mooij (1973), Ti$_{1-x}$Al$_x$ alloys**
  - $x = 0$ **(Pure Ti) is doing what it should at low** T**, but at high** T **seems to be "saturating"**
  - **For large** x**,** $\frac{\mathrm{d}\rho}{\mathrm{d}T}$ **is** *negative!!!*
  - **All the resistivities are** *tending to a roughly equal saturation value***!**
  - **The saturation resisitivity** *2* **orders of magnitude higher than usual metallic values...**



- **Note that these are binary** *alloys***...crystals with a random placing of Ti and Al ions! Electrons see a "highly disorderd" potential!**

There's more!

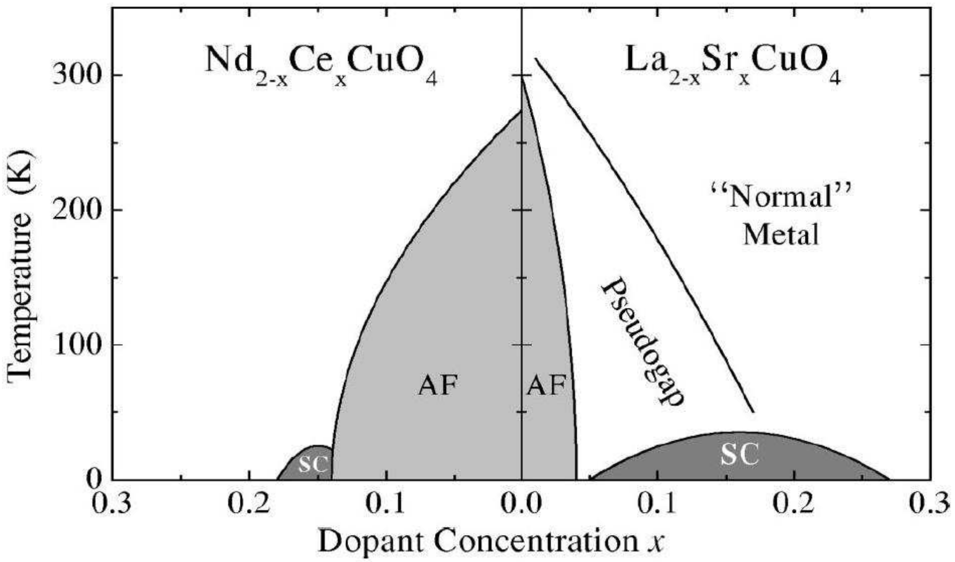
- **Mooij found that the low temperature $\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}T} \equiv \alpha$ is related to the "residual resistivity"**

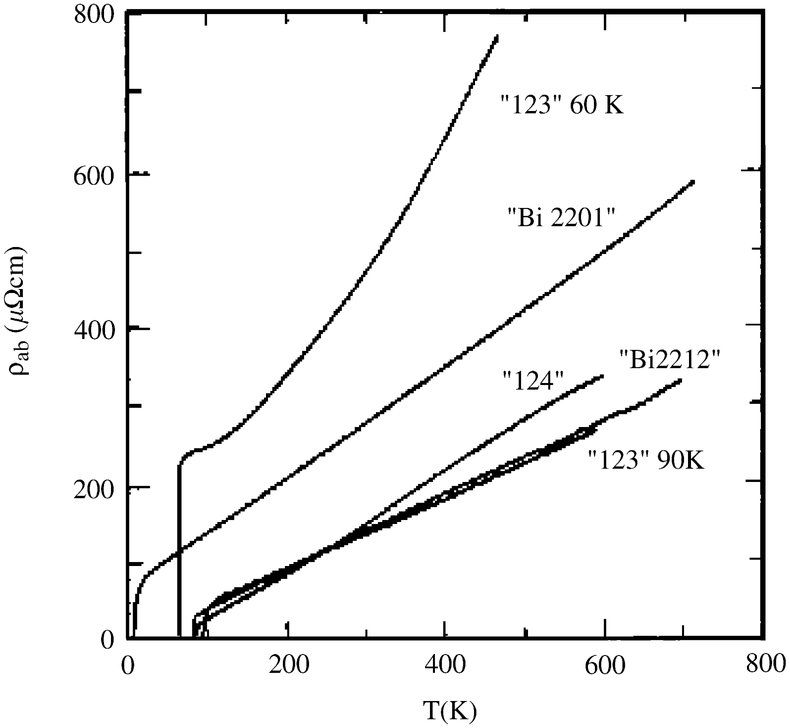


- **This *is* magic! The key resistivity is about 100 $\mu\Omega$-cm! If low T resistivity exceeds this value, then strange things happen...**
- **Most interestingly, similar stuff is seen in other disordered alloys!**

How can we forget the Cuprates?

- **Resistivity in high T_c normal state**





- **What is ($are?$) the puzzle(s) here?**

Story so far...

- **The Bloch-Boltzmann theory is highly successful in explaining resistivities of elemental metals containing a small concentration of impurities**
- **Experiments on disordered alloys suggest**
  - **Possibility of negative** $\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}T}$
  - **Correlated with low temperature resistivity; if low** T **resistivity** $\gtrsim$ **100** $\mu\Omega$**-cm, we have negative** $\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}T}$
- **How do we understand this?**
- **Before we get to the answer, we need to understand resistivity and its relation to other response functions**

Meaning of Standard Formula for Conductivity

- **The "standard result" for conductivity**

$$\sigma = \frac{ne^2\tau}{m}$$

(all symbols have usual meanings)

- **Meaning of τ – sharper meaning for $\frac{1}{\tau}$ – $\frac{1}{\tau}$ is the number of collisions undergone by the electron per unit time...**
- **Idea: Think of an electron to be in a $\boldsymbol{k}$ state at time $t = 0$, then τ is the life-time of such a state**
- **Also, the electron** *does not remember* **past collisions!**

Meaning of τ

- **To see that τ is the life-time, ask what is the probability $P(t)$ that the electron is still in the state $\boldsymbol{k}$ for $t \lesssim \tau$...**
- **The probability that the electron did not undergo a collision in an infinitesimal time ϵ is $\left(1 - \dfrac{\epsilon}{\tau}\right)$...Thus,**

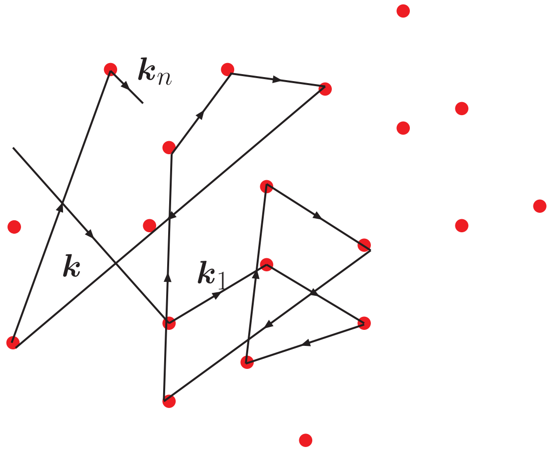
$$P(t) = \lim_{N\to\infty} \left(1 - \frac{1}{\tau}\frac{t}{N}\right)^N = e^{-t/\tau}$$

- **This precisely connects up with our earlier statement: An electron placed in a Bloch state $\boldsymbol{k}$ evolves according to $\psi(t) \sim \psi_{\boldsymbol{k}} e^{-i\varepsilon(\boldsymbol{k})t - \frac{t}{2\tau_{\boldsymbol{k}}}}$; "lifetime" is $\tau_{\boldsymbol{k}} \sim \tau$!**

Caveat: Note however that τ is Drude formula is the *transportation lifetime*

So what is the electron doing?

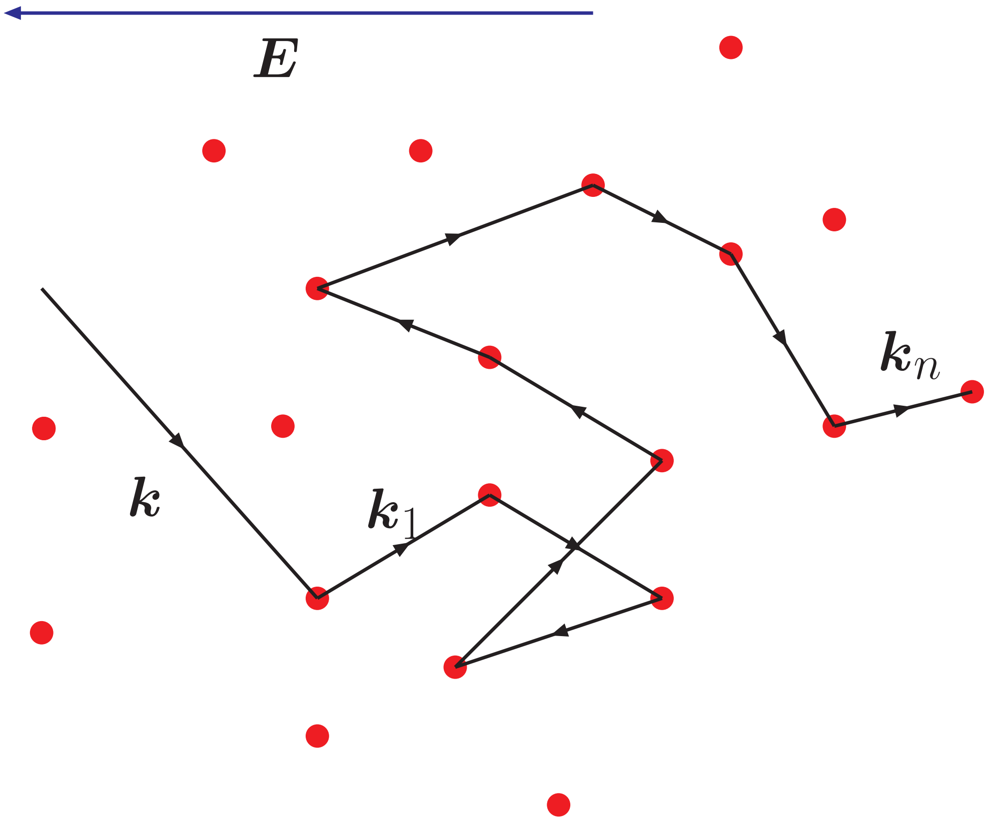
- A collision puts the electron in a different Bloch state (of same energy, discussion restricted to *impurity scattering*)...classically, simply changed direction...
- A moment's reflection tells us that the electron is *RANDOM WALKING*



- Since the average velocity of electrons is v_F, the *mean free path* of electrons is $\ell = v_F \tau$ **Caveat: Note that this, in general, is *not* the "average spacing between impurities"**

So what is the electron doing in an E-field?

- **The electron is random walking and** *drifting***!**



- **Develops a** *drift velocity* $|\boldsymbol{v}_d| \sim \tau |\boldsymbol{E}|$

Consequences of Random Walk

- **So what? How is this different from a *free* gas (no scattering) where electrons propagate with only Pauli to respect?**
- **Key question: Suppose we create a very long wavelength (compared to inverse Fermi vector) density disturbance (without changing the total number of electrons) in the electron gas...what difference does the scattering (dirt) make to this**
- **In the free gas, there is no mechanism to "relax" this density wave...and the system will simply "do some dynamics"...can never attain a uniform density back again! We will call this the Free Fermi Fixed Point (more later...)!**

Consequences of Random Walk

- **What happens in a dirty gas?**
- **Particles, doing random walk, will** *DIFFUSE***! Well known elementary result!**
- **Punch line: Presence of** *weak* **disorder will give us a** *qualitatively new state***... a state with a** *diffusive density mode* **(not present in the free)...this state is the Diffusive Fixed Point**
- **Note that there will be no qualitative differences in the** *thermodynamic properties* **of DFP and FFFP!**
- **Why the word** *weak***?**
- **What is** *"not weak"* **is the question that we will address in great detail...**

Consequences of Random Walk: Diffusion

- **At the diffusive fixed point (DFP), we are guaranteed that density fluctuations $n(\boldsymbol{r},t)$ will be governed by**
  - **Continuity equation (conservation law) – always holds (J particle current)**

$$\partial_t n + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0$$

  - **Diffusive constitutive (Fick's) law (property of DFP)**

$$\boldsymbol{J} = -D\boldsymbol{\nabla} n$$

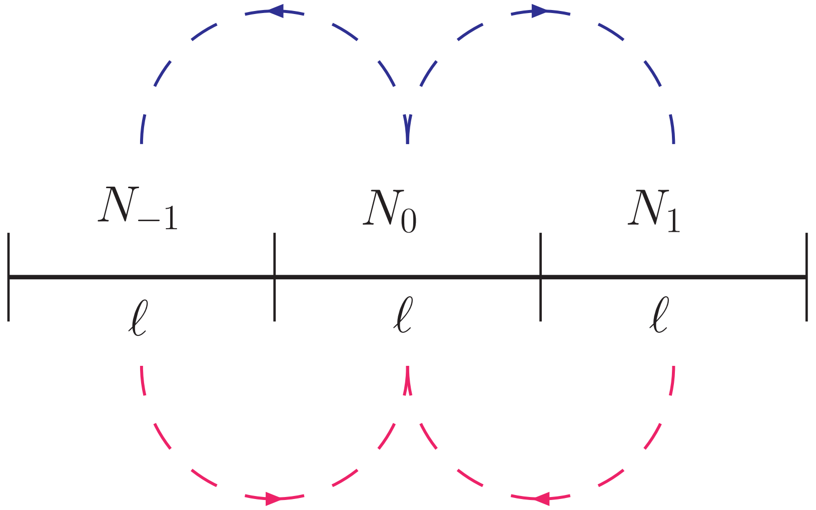
D **is the** *diffusion coefficient*...**property of the DFP**

Caveat: A more "correct" form is $\boldsymbol{J}(\boldsymbol{q},\omega) = -D(\boldsymbol{q},\omega)i\boldsymbol{q}n(\boldsymbol{q},\omega)$

- **What determines D?**

The Diffusion Coefficient

- **Consider dicing up space into cubes of size ℓ^d (in d dimensions)**
- **Take three adjacent "cubes" (in 1-d) called -1, 0, 1 with N_{-1}, N_0 and N_1 particles at time $t = 0$**



- **At time $t = \tau$, the number of particles in the 0 cube is $\frac{1}{2}(N_{-1} + N_{+1})$**

The Diffusion Coefficient

- Change in particle number $\Delta N_0 = \frac{1}{2}(N_{-1} + N_{+1} - 2N_0)$

$$\underbrace{\frac{\Delta N_0}{\tau}}_{\partial_t n} = \underbrace{\frac{\ell^2}{2\tau}}_{D} \underbrace{\frac{(N_{-1} + N_{+1} - 2N_0)}{\ell^2}}_{\nabla^2 n}$$

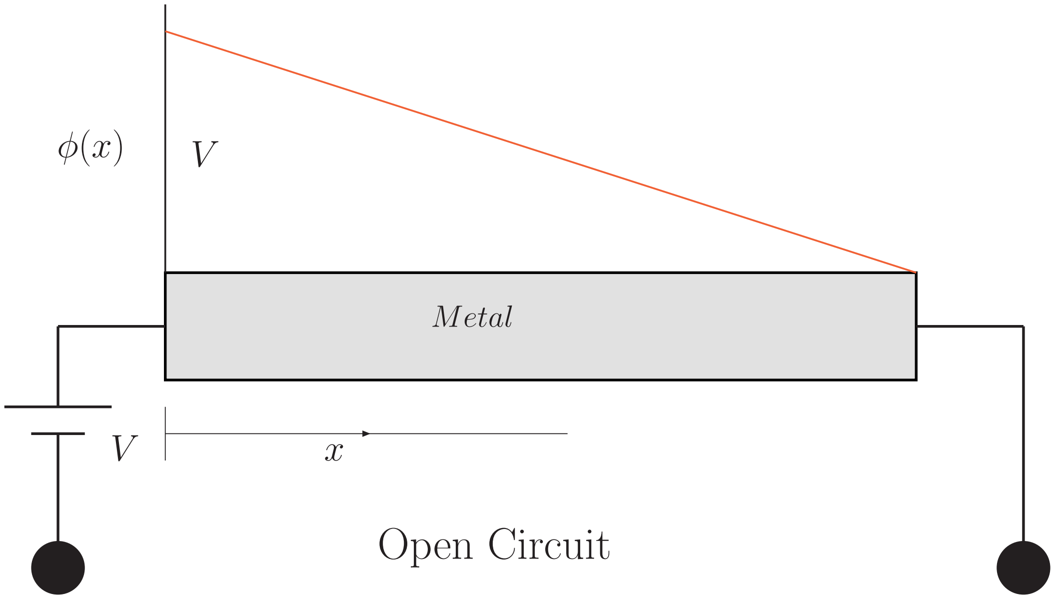
- If we do this correctly in 3D, we will get

$$D = \frac{1}{3} v_F^2 \tau$$

- We see that D is linearly related to τ...this is not the first time this has happened! Note that the conductivity is also linear in τ!!!
- Are σ and D related?

Diffusion Coefficient and Conductivity

- **Here is an experiment**



- **The "undisturbed" metal has electron density n and chemical potential μ**
- **The battery generates an electric potential $\phi(x)$**
- **Open circuit...there is not current flowing through the circuit!**

Diffusion Coefficient and Conductivity

- **Since there is a potential gardient, there is obviously an electric current $j = -\sigma \partial_x \phi$...it is not zero! Something fishy?**
- **No...there is another contribution to the current coming from diffusion...**
- **The chemical potential varies in space $\mu(x) = \mu + e\phi(x)$ resulting is a density variation given by** $n(x) \approx n + \frac{\partial n}{\partial \mu} e\phi$
- **Diffusive particle current $J = -D\frac{\partial n}{\partial \mu} e \partial_x \phi$...which contributes to a** *diffusive electric current* $j_D = -eJ = D\frac{\partial n}{\partial \mu} e^2 \partial_x \phi$

Diffusion Coefficient and Conductivity

- **Now the total current must vanish** $j + j_D = 0$
- **We get (**$g(\mu)$ **– density of states at the chemical potential)**

$$\sigma = e^2 \frac{\partial n}{\partial \mu} D = e^2 g(\mu) D$$

This is the famed *Einstein relation*...**diffusion and electrical conduction are** *closely* **related!**

- **This is a result of particle number conservation...and hence applicable in** *any* **system!!**
- **At the DFP, finite diffusion coefficient implies a finite conductivity!**
- **Our next step is to derive this relation from a formal point of view...**

Einstein Relation: Plan of Derivation

- **Understand "charge susceptibility"** $\chi(\boldsymbol{q}, \omega)$
- **Conservation law : relationship between** χ **and** σ
- **Relationship between** χ **and diffusion constant - exploration of relaxation of density disturbance**
- **...which gives the Einstein relation**

Density Response: "Charge Susceptibility"

- **Suppose we vary the potential (about the homogeneous value) as $\mu(\boldsymbol{r}, t)$, then as usual the density response (excess over the homogeneous value) is given by**

$$n(\boldsymbol{r}, t) = \int \mathrm{d}^d\boldsymbol{r}' \int \mathrm{d}t' \, \chi(\boldsymbol{r}-\boldsymbol{r}', t-t')\mu(\boldsymbol{r}', t')$$

where ($\hat{n}$ – "excess density" operator)

$$\chi(\boldsymbol{r}-\boldsymbol{r}', t-t') = -i\theta(t-t') \underbrace{\langle[\hat{n}(\boldsymbol{r}-\boldsymbol{r}', t-t'), \hat{n}^\dagger(\boldsymbol{0}, 0)]\rangle}_{\widetilde{\chi}(\boldsymbol{r}-\boldsymbol{r}', t-t')}$$

- **In Fourier language**

$$n(\boldsymbol{q}, \omega) = \chi(\boldsymbol{q}, \omega)\mu(\boldsymbol{q}, \omega)$$

- **What can we say about χ without solving anything?**

Density Response: "Charge Susceptibility"

- **Consider a** *time independent* **static potential perturbation $\mu(\boldsymbol{q})$ that was statically turned on at $t = -\infty$**
- **The response function is $\chi(\boldsymbol{q}, 0)$...we can calculate this using standard statistical mechanics formulae ($\hat{N}(\boldsymbol{q})$ is the full number operator, not excess)**

$$n(q) = \langle \hat{N}(\boldsymbol{q}) \rangle - \langle \hat{N}(\boldsymbol{q}) \rangle_0 = \underbrace{\chi(\boldsymbol{q}, 0)}_{\chi^s(\boldsymbol{q})} \mu(\boldsymbol{q})$$

$$\lim_{\boldsymbol{q}\to\boldsymbol{0}} \chi(\boldsymbol{q}, 0) = \frac{\partial n}{\partial \mu} = g(\mu)$$

- **Conclusion**

$$\lim_{\boldsymbol{q}\to\boldsymbol{0}} \lim_{\omega\to 0} \chi(\boldsymbol{q}, \omega) = g(\mu)$$

Density Response: "Charge Susceptibility"

- **Now consider a time dependent disturbance of the potential...this will cause particles to run around in the system...this "running around" preserves particle number**
- **This would imply that at all times t (remember n is excess)**

$$\begin{aligned}\int \mathrm{d}^d\boldsymbol{r}\, n(\boldsymbol{r},t) = 0 \quad &\Longrightarrow \quad \int \mathrm{d}^d\boldsymbol{r}\, \chi(\boldsymbol{r},t) = 0 \\ &\Longrightarrow \quad \chi(\boldsymbol{q} = \boldsymbol{0}, \omega) = 0\end{aligned}$$

- **We thus see that the response function χ is not analytic near $\boldsymbol{q} = 0$ and $\omega = 0$...**
- **Note that these results are very general...do not depend on the Hamiltonian**

Density Response: "Charge Susceptibility"

- **We know** $\chi(\boldsymbol{r}-\boldsymbol{r}', t-t') = -i\theta(t-t')\widetilde{\chi}(\boldsymbol{r}-\boldsymbol{r}', t-t')$
- **Recast this as** **Recall:** $\int_{-\infty}^{\infty} \mathrm{d}t\; \theta(\pm t)\; e^{i(\omega \pm i\eta - \epsilon)t} = \frac{\pm i}{\omega \pm i\eta - \epsilon}$

$$\chi(\boldsymbol{q}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\widetilde{\chi}(\boldsymbol{q}, \omega')}{\omega^{+} - \omega'}$$

- **Note that** $\widetilde{\chi}(\boldsymbol{q}, \omega)$ **is real** **Exercise: Show this!**
- **Now it is easy to see that** **Recall:** $\frac{1}{x^{+}} = P\frac{1}{x} - i\pi\delta(x)$

$$\begin{aligned} \Re(\chi(\boldsymbol{q}, \omega)) = \chi'(\boldsymbol{q}, \omega) &= \frac{1}{2\pi} P\!\!\!\!\!\int_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\widetilde{\chi}(\boldsymbol{q}, \omega')}{\omega - \omega'} \\ \Im(\chi(\boldsymbol{q}, \omega)) = \chi''(\boldsymbol{q}, \omega) &= -\frac{1}{2}\widetilde{\chi}(\boldsymbol{q}, \omega) \end{aligned}$$

Relaxation of a "Static" Perturbation

- **Immediate goal: Connect χ to D**
- **To do this we ask the following question: Suppose a static potential $\mu(\boldsymbol{q})$ is turned on at $t = -\infty$...a static density response will develop in the system and is given by $n^s(\boldsymbol{q}) = \chi^s(\boldsymbol{q})\mu(\boldsymbol{q})$...**
- **At time $t = 0$ we *switch off* the external potential...what happens?**
- **The density perturbation $n^s(\boldsymbol{q})$ will relax... Question: Will this happen in a free gas?**
- **The relaxation function (Kubo function) is defined as**

$$n(\boldsymbol{q}, t) = \Phi(\boldsymbol{q}, t)\mu(\boldsymbol{q}), \qquad t > 0$$

- **How is $\Phi(\boldsymbol{q}, t)$ related to $\chi(\boldsymbol{q}, t)$?**

Kubo Relaxation Function

- **How is $\Phi(\boldsymbol{q}, t)$ related to $\chi(\boldsymbol{q}, t)$?**
- **Clearly**

$$\Phi(\boldsymbol{q}, t) = \int_{-\infty}^{0} \mathrm{d}t' \; \chi(\boldsymbol{q}, t - t')$$

- **Going over to Fourier language**

$$\begin{aligned} \Phi(\boldsymbol{q}, \omega) &= \int_{-\infty}^{\infty} \mathrm{d}t \; e^{i\omega t} \int_{-\infty}^{0} \mathrm{d}t' \; - i\theta(t - t')\widetilde{\chi}(\boldsymbol{q}, t - t') \\ &= \frac{1}{i\omega^{+}} \left(\chi(\boldsymbol{q}, \omega) - \chi^{s}(\boldsymbol{q}) \right) \end{aligned}$$

Exercise: Show this

Diffusive Relaxation

- **Based on earlier arguments we can obtain another expression for Φ at the DFP**
- **At the DFP, density relaxes via**

$$\partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0, \qquad \boldsymbol{J}(\boldsymbol{q},\omega) = -D(\boldsymbol{q},\omega) i\boldsymbol{q} n(\boldsymbol{q},\omega)$$

- **It is immediate (for $D(\boldsymbol{q},\omega) = D$) that** $n(\boldsymbol{q},t) = n^s(\boldsymbol{q}) e^{-Dq^2 t}, \quad t > 0$**...diffusive relaxation**
- **Defining $\tilde{n}(\boldsymbol{q},\omega) = \int_{-\infty}^{\infty} \mathrm{d}t e^{i\omega t} \theta(t) n(\boldsymbol{q},t)$, we see that for**

$$\tilde{n}(\boldsymbol{q},\omega) = \frac{i n^s(\boldsymbol{q})}{\omega + iDq^2} \Longrightarrow \Phi(\boldsymbol{q},\omega) = \frac{i\chi^s(\boldsymbol{q})}{\omega + iDq^2}$$

- **Φ has a *pole*...diffusion pole!**

Diffusive Relaxation

- **We now have two expressions for Φ...**
- **The first one is from general considerations of the response function**
- **The second one is special to the DFP**
- **They must be equal**

$$\begin{aligned}\Phi(\boldsymbol{q},\omega) &= \frac{1}{i\omega^+}\left(\chi(\boldsymbol{q},\omega)-\chi^s(\boldsymbol{q})\right)=\frac{i\chi^s(\boldsymbol{q})}{\omega+iDq^2}\\ \Longrightarrow \chi(\boldsymbol{q},\omega) &= \frac{iDq^2\chi^s(\boldsymbol{q})}{\omega+iDq^2}\end{aligned}$$

Diffusive Relaxation

- **We see that at DFP, the density response function has the special form...**
- **Lets check if things are alright**

$$\lim_{\omega \to 0} \chi(\boldsymbol{q}, \omega) = \chi^s(\boldsymbol{q}), \qquad \lim_{\boldsymbol{q} \to \boldsymbol{0}} \chi(\boldsymbol{q}, \omega) = 0$$

...that's good!

- **Note that the density response function has a pole in the lower half plane...this is a characteristic feature of the DFP...called the** *diffuson* **pole.**
- **Now Einstein is in sight!!**

General Electro-Magnetic Response

- **Drive the system by an electromagnetic field $A_\mu(\boldsymbol{r},t)$...($A_0 \equiv \phi$, and $A_i \equiv \boldsymbol{A}$ etc...)** Caveat: I may not be using the standard notation; also we will respect up-down indices since it will help us to express equations in a compact manner
- **Response function is j^μ (again, $j^0 \equiv -en$ – charge density, and $j^i \equiv \boldsymbol{j}$**
- **Most general linear response $K^{\mu\nu}$**

$$\begin{aligned} j^\mu(\boldsymbol{r},t) &= \int \mathrm{d}^3\boldsymbol{r}' \int \mathrm{d}t'\; K^{\mu\nu}(\boldsymbol{r}-\boldsymbol{r}',t-t')A_\nu(\boldsymbol{r}',t') \\ j^\mu(\boldsymbol{q},\omega) &= K^{\mu\nu}(\boldsymbol{q},\omega)A_\nu(\boldsymbol{q},\omega) \end{aligned}$$

- **The general response seems to be a 4×4 tensor...are they all independent?...more importantly how is $K^{\mu\nu}$ related to χ and σ?**

General Electro-Magnetic Response

- **First and key point...**$K^{\mu\nu}$ *must be gauge invariant*...**so if we replace** $A_\mu \Longrightarrow A_\mu + \partial_\mu \zeta$ **for any** $\zeta(\boldsymbol{r}, t)$, **then** j^μ **should not change...this gives us a key condition that**

$$K^{\mu\nu} q_\nu = 0$$

- **Similarly, since particle number is conserved, we need**

$$\partial_\mu j^\mu = 0 \Longrightarrow q_\mu K^{\mu\nu} = 0$$

Recall: Up-down contraction involves $g^{\mu\nu} = \mathbf{Diag}(-1, 1, 1, 1)$, **also** $q^0 = \omega$, **and** $q^i \equiv \boldsymbol{q}$

General Electro-Magnetic Response

- **Let us connect K to χ and σ...**
- **Now consider a case with only the potential part...$A_0 = \phi, ..., A_i = 0$**
- **Then**

$$j^0 = K^{00}\phi \Longrightarrow -en = K^{00}\frac{\mu}{e} \Longrightarrow K^{00} = -e^2\chi$$

- **Similarly,**

$$j^i = K^{i0}\phi \Longrightarrow j^i = \frac{1}{iq_i}K^{i0}E_i \Longrightarrow K^{i0} = iq_i\sigma$$

General Electro-Magnetic Response

- **Applying the conservation law**

$$\begin{aligned} q_\mu K^{\mu\nu} = 0 \Longrightarrow \omega e^2 \chi(\boldsymbol{q},\omega) + i q^2 \sigma &= 0 \\ \sigma(\boldsymbol{q},\omega) &= -e^2 \frac{i\omega}{q^2} \chi(\boldsymbol{q},\omega) \end{aligned}$$

Note that this is a general result (independent of the Haimiltonian)...

- **At DFP, we have**

$$\sigma(\boldsymbol{q},\omega) = -e^2 \frac{i\omega}{q^2} \frac{iDq^2 \chi^s(\boldsymbol{q})}{\omega + iDq^2} = \frac{e^2 \omega \chi^s(\boldsymbol{q})}{\omega + iDq^2}$$

Thus at the DFP, the conductivity is completely determined by diffusion!

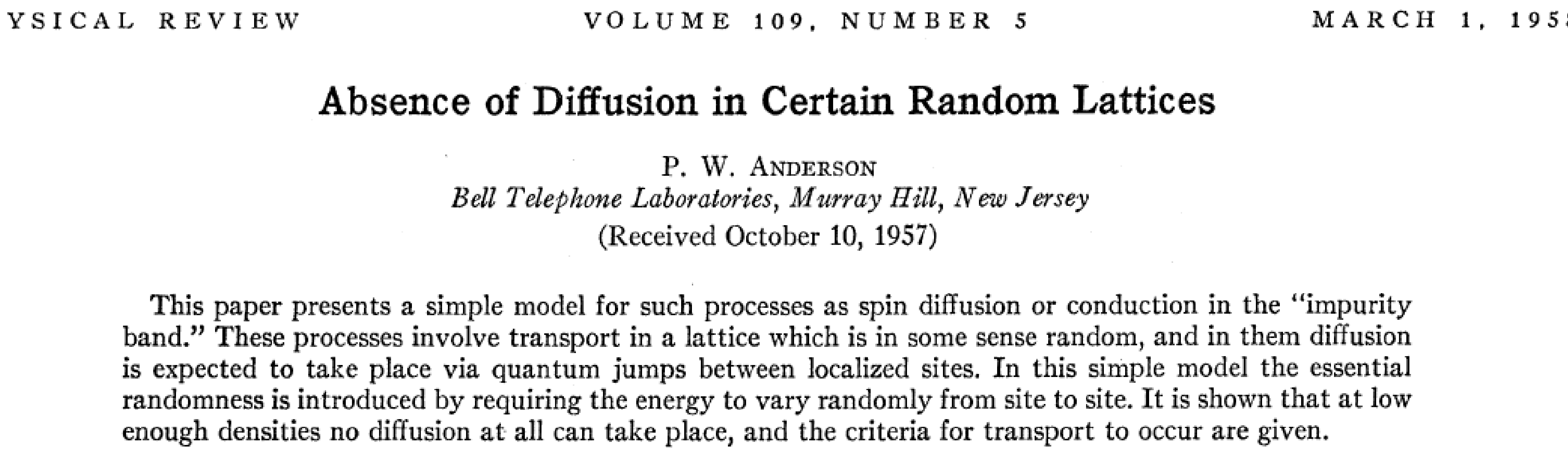
And Einstein Appears!

- **We immediately see that the DC conductivity σ is realted to the diffusion constant D via**

$$\sigma = e^2 g(\mu) D$$

- **This is a rather general feature of the DFP...transport properties can all be finally related to D Exercise: How about thermal conductivity?..can you see Widemann-Franz?**
- **This is really a consequence of the underlying conservation laws...**
- **When will σ go to zero? The case interesting for us is when D vanishes....i. e., "absence of diffusion" which takes us back to 1958!**
- **Punch line: When disorder is "not weak", i. e., "strong enough"...D will vanish and we get an** *Anderson insulator*...**i. e., $g(\mu) \neq 0$, but is not an electrical conductor!**

...followed by Anderson!



YSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 195

Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON
Bell Telephone Laboratories, Murray Hill, New Jersey

(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

- **This is the beginning of the field...**

Our Plan

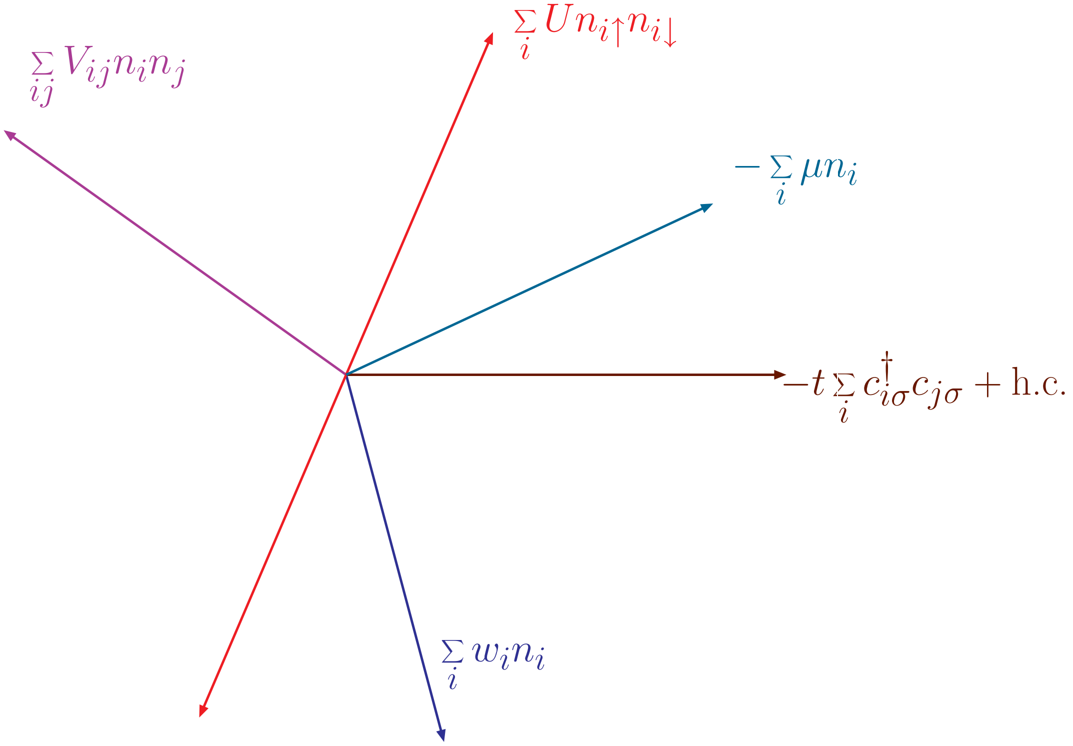
- **Introduce models, ideas of extended localized states etc..**
- **Discuss “Phase Diagrams”...this will be the summary of the story**
- **Scaling theory of localization...**
- **Diagrammatic “derivation” of scaling theory**
- **Replica field theory...non-linear sigma model approaches and RG**

The “Pancha Bhutas”

- **We will focus on the** *electronic* **sector (no phonons), and treat effects of things likes phonons by means of** *effective coupling constants*
- **The “Pancha Bhutas”**
    - **Kinetic energy (t)**
    - **Chemical potential (μ)**
    - **Long range repulsion (V)**
    - **On-site correlation (U, can be repulsive or attractive)**
    - **Disorder (w)**
- **The “Five Elements” – everything we “see” arises from these**

The "Pancha-Bhuta-Chitra"

- **The phase diagram (at $T = 0$, and $T > 0$), the "Pancha-Bhuta-Chitra" (PBC), is the goal of HCMP!**



- **Our goal: To study the phase diagram in the t, μ, w "plane"!**

Models of Disordered Non-interacting Electrons

- We do not need to worry about spin
- Many different models are available
  - Free gas with random distribution of scatterers (Edwards model)
  - Electrons on a lattice with random on-site potential (Anderson model)
  - Electrons on a lattice with two values of on-site potential which are distributed randomly (Alloy model)
  - Electrons on a lattice with hopping amplitude which is an average value plus a random perturbation
  - ...
- We will work with the first two...

Edwards Model (EM)

- **Free electrons moving in a background of random scatters**

$$\mathcal{H} = \int \mathrm{d}^d\boldsymbol{r}\, \psi^\dagger(\boldsymbol{r}) \left(-\frac{1}{2m}\nabla^2 - \mu + V(\boldsymbol{r}) \right) \psi(\boldsymbol{r})$$

(we use ψ for free electron operators) where

$$V(\boldsymbol{r}) = \sum_i v(\boldsymbol{r} - \boldsymbol{R}_i)$$

i runs over the impurities, v is the potential due to a single impurity (assume that all impurities are of the same type), $\boldsymbol{R}_i$ is the position of the ith impurity

- **Impurities are randomly and uniformly distributed...i. e., $\boldsymbol{R}_i$ is a random variable with can take any vector value in our volume with equal probability**
- **Parameters of the model:**
  - **Density n ($\equiv \mu$) of electrons**
  - **Density n_{imp} of impurities**
  - **Scattering potential $v(\boldsymbol{r})$ (parametrize this by one number later)**

Anderson Model (AM)

- **This is the model introduced by Anderson in 1958**
- **Electrons move on a lattice**
- **At each site there is a random on-site potential w_i which is usually taken to be uniformly distributed between $-W$ and W**

$$-t\sum_{ij}\left(c_i^\dagger c_j + \textbf{h. c.}\right) + \sum_i \left(w_i - \mu\right) n_i$$

- **Parameters : μ/t and W/t**
-
- **We shall use Edwards and Anderson models to gain an understanding of the disorder problem**

Disorder Preliminaries

- **Note that we have introduced a Hamiltonian that contains many random disorder parameters (e.g., w_i in the AM)**
- **For** *each* **realization of the disorder w_i, we get a** *different* **Hamiltonian $\mathcal{H}_w$! What we therefore have is an** *ensemble* **of Hamiltonians!**
- **Key Question: What are we observing when we do measurements on a particular sample? Will the answer be "sample dependent"?**
- **The answers are subtle...**

Disorder Preliminaries

- **Suppose we are a realization of the disorder...then if we do a measurement of the observable $\mathcal{A}$ in a system in thermal equilibrium, we obtain**

  $$\langle \mathcal{A} \rangle_w = \mathrm{Tr}(\varrho_w \mathcal{A}) = \sum_\alpha \varrho_{\alpha_w} \langle \alpha_w | \mathcal{A} | \alpha_w \rangle, \qquad \varrho_w = \frac{e^{-\beta \mathcal{H}_w}}{\mathrm{Tr} e^{-\beta \mathcal{H}_w}}$$

  $|\alpha_w\rangle$ **are the eigenstates of** $\mathcal{H}_w$

- **Note that this formula involves** *two* **types over averages: first,** $\langle \alpha_w | \mathcal{A} | \alpha_w \rangle$ **is a quantum average, the second is the averaging over the** *thermal probability* **distribution and is accomplished by** $\mathrm{Tr}(\varrho \ldots)$**...The final observed value is a** *thermal average* **of** *quantum averages***...a very "average" thing!!**

- **But** $\langle \mathcal{A} \rangle_w$ **depends on the realization of disorder!!**

Disorder Preliminaries

- **So, what are we measuring? Would it be a "disorder averaged" quantity?**
- **The disorder averaged quantity**

$$\langle\!\langle \mathcal{A} \rangle\!\rangle = [\![\langle \mathcal{A} \rangle_w]\!] = \sum_w P(w) \langle \mathcal{A} \rangle_w$$

Notation: $\langle \ldots \rangle$ – thermal average, $[\![\ldots]\!]$ – disorder average, $\langle\!\langle \ldots \rangle\!\rangle$ – disorder averaged thermal average

- **When do we expect the measured quantity to be $\langle\!\langle \mathcal{A} \rangle\!\rangle$? Naturally, if a system is "very large", then different parts of the system will behave as "separate members of the disorder ensemble" and the measured quantities will correspond to disorder averages**
- **Natural question: what is "very large"?**

Disorder Preliminaries

- **Suppose we consider a system of size "N", then we can calculate the disorder average of any quantity Q and its variance defined as**

$$[\![Q_w]\!]_N = \sum_w P(w) Q_{wN}, \quad V_N^Q = [\![Q_w^2]\!]_N - [\![Q_w]\!]_N^2$$

- **Now** *if*

$$R_N^Q = \frac{V_N^Q}{[\![Q_w]\!]_N^2} \sim \frac{1}{N}$$

then we say that Q is a *self-averaging quantity***!**

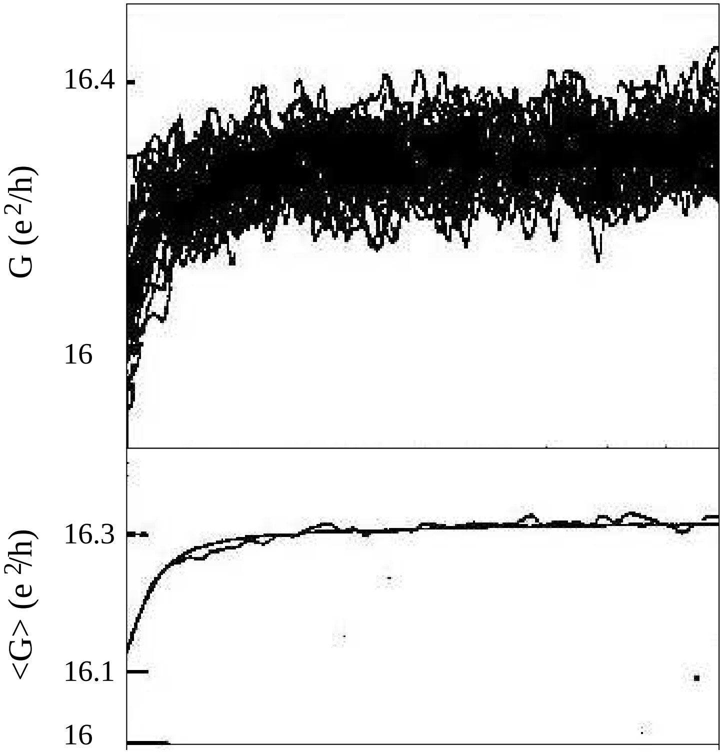
- **If a quantity Q is self averaging, then for large systems what we measure will be the disorder average!**

Disorder Preliminaries

- **The property of self averaging depends on** *the quantity* **and** *the system*...Caveat: Concepts such as strong and weak self averaging are around
- **In the case of bulk metals with disorder, we can be sure that diffusion coefficient (and hence conductivity) are self averaging quantities**
- **Certain critical systems with disorder are known to be non-self averaging! (This is topic of current research)**
- **What we measure may not be a disorder averaged quantity due to the small size of the systems....,i.e.,** N **is not large enough to kill the statistical variations,...such systems are called** *mesoscopic*! **They are expected to show** *fluctuations* **between one sample and another...**

Disorder Preliminaries

- **This is indeed seen in experiments: conductance of SiGaAs wires (Malliy and Sanquer 1992)**
  - **Conductance depends on magnetic field**
  - **Different curves are obtained when the sample is thermally cycled...heated and cooled back to the same (low) measurement temperature...this makes the disorder "move around"!**
  - **The bottom curve shows the average over many such cycles**



B **(0 T to 0.15 T)**

- **We shall focus only on** *self averaging systems!*

What is the difficulty?

- **The two models that we have introduced are quadratic in fermion operators...so you might wonder what the difficulty is!**
- **What we want to calculate is**

$$\langle\!\langle \mathcal{A} \rangle\!\rangle = [\![\frac{\mathrm{Tr} \varrho_w \mathcal{A}}{Z_w}]\!], \quad Z_w = \mathrm{Tr} \varrho_w$$

- **This requires calculation of energy eigenstates for** *each realization* **of** w**!!! This** *is* **the difficulty**
- **There are many tricks to handle this...replica trick, super-symmetry approach, Keldysh formalism etc.. we shall see the replica trick**
- **But before we do the technical stuff, we will discuss the physics..**

Physics of the Edwards Model

- **Consider the Edwards model with $n_{imp} \ll n$ (parts per million)**
- **Also consider that the scattering potential $v(\boldsymbol{r})$ is a delta potential...$v(\boldsymbol{r}) = v\delta(\boldsymbol{r})$...assume that $v \ll k_F^3 \mu$...**
- **This would constitute a "weak disorder" problem...and the conductivity would be given by the Drudé formula with**

$$\frac{1}{\tau} \sim n_{imp} v^2$$

- **What would "not weak disorder" mean...to understand this let us take a deeper look at the Drudé formula...**

Closer Look at the Drudé Formula

- The Drude formula in d spatial dimensions

$$\sigma = \frac{ne^2\tau}{m} = \frac{\overbrace{(C_d k_F^d)}^{n}\, e^2 \overbrace{\ell}^{v_F\tau}}{m\underbrace{\left(\frac{\hbar k_F}{m}\right)}_{v_F}} = \frac{e^2}{\hbar} C_d k_F^{d-2}\,(k_F\ell)$$

Question: What is $\frac{e^2}{\hbar}$? (Ans: 0.25 milli-Ω^{-1})

- Suddenly we realize what is "weak"...the disorder is "weak" when the mean free path is much larger than the inverse Fermi vector..., i. e., when $k_F\ell \gg 1$...in this case then we have nothing to do...

- But what if $k_F\ell \sim 1$??...can $k_F\ell \ll 1$???? If $k_F\ell \sim 1$ the electron is moving only a distance of order of the inter-electron spacing ($\sim$ lattice spacing) between collisions...thus is essentially *not* moving!! There *is trouble if the mean free path is less than the de Broglie wavelength of the electron*!! Thus if the disorder is "not weak" the electron *is unable to random walk*...absence of diffusion!! $k_F\ell \sim 1$ corresponds to the Ioffe-Regel limit!

- We will discuss $d = 3$ and $d = 2$ separately...but not $d = 1$!

Ioffe-Regel Conductivity in $d = 3$

- **In $d = 3$ ($a \sim$ lattice parameter, $k_F = \frac{2\pi}{a}$)**

$$\sigma_{IR} = \frac{e^2}{\hbar}\frac{1}{3\pi^2}k_F\underbrace{(k_F\ell)}_{\sim 1} = \frac{e^2}{\hbar}\frac{2}{3\pi}\frac{1}{a}$$

- **The resistivity at such strong disorder is**

$$\rho_{IR} = \underbrace{\frac{\hbar}{e^2}}_{4000\Omega}\underbrace{\frac{3\pi}{2}}_{4}\underbrace{a}_{2.5\mathring{A}} \sim 400\mu\Omega\text{-}\mathbf{cm}$$

this is definitely an over-estimate, but we are quite close to the low temperature resistivity of 100 $\mu\Omega$-cm where trouble beings in the Mooij experiments...

Ioffe-Regel Conductivity in $d = 3$

- **In the Edwards model, if τ becomes small (either by a large n_i or by a large v (strength of the potential)), then we will get into the regime where $k_F \ell \sim 1$**
- **We thus see *why* strong disorder causes trouble...the key point is that electron mean free path becomes too small for it to be able to diffuse!!**
- **Thus as the disorder is made stronger, the system goes to a "new phase" where *there is no diffusion*...we will call this the Anderson Fixed Point (AFP)...**
- **Question: What is the nature of the "phase transition" between these two "phases"...i.e., the "diffusive phase" and "Anderson phase"?**
- **Mott took Drudé seriously and concluded that metal must have a minimum conductivity...the famous Mott minimum!! Accoding to Mott the transition from the "diffusive phase" to the "Anderson phase" is "first order", i.e., accompanied by a discontinuous jump in the conductivity (This has turned out to be wrong, but it is a beautiful idea nevertheless...)**
- **In the remainder of these discussions we will understand the nature of this quantum phase transition...but before that lets see what 2d has in store for us..**

Ioffe-Regel Conductivity in $d = 2$

- **In $d = 2$ ($a \sim$ lattice parameter, $k_F = \frac{2\pi}{a}$)**

$$\sigma_{IR} = \frac{e^2}{\hbar}\frac{1}{\pi}k_F^0 \underbrace{(k_F\ell)}_{\sim 1} = \frac{e^2}{\hbar}\frac{2}{\pi}$$

- **The resistivity at such strong disorder is**

$$\rho_{IR} = \underbrace{\frac{\hbar}{e^2}}_{4000\Omega}\frac{1}{2} \sim 20\mathbf{k}\Omega$$

...this is amazing! The Ioffe-Regel resistivity turns out to be a universal number (independent of k_F) in 2d...

- **Thus, if we see data that shows values of resistivity (or sheet resistance, as it is called in 2d), then we know we are in the strong disorder regime...**

Summary of Discussions

- **Expect Drudé formula to hold for weak disorder**
- **"Weak" means $k_F \ell \gg 1$**
- **When $k_F \ell \to 1$, resistivity will increase...the typical order of magnitude when $k_F \ell \to 1$ is called the Ioffe-Regel limit ($\sim$100 $\mu\Omega$-cm (3d), $\sim$10 KΩ (2d))...**
- **In any experimental system if we see low temperature resistivity greater than the IR limit it is suggestive**
- **As the strength of disorder increases, we will have a "new phase" where the mean free path is so small that it has no meaning, i. e., the electron will stop diffusing**
- **Key question: Is there a "phase transition" as a function of disorder strength? What is the nature of the transition if it exists?**

“Short Term” Plan

- **Understand the *physics* of the Anderson transition...yes, it exists...it is also called as the *localization transition*...**
- **Understand what is happening in the Anderson model as a function of disorder...this will help us understand why we use the phrase *localization*!**
- **Discuss scaling theory of localization, and show that there is indeed a transition (this will turn out to be true only in 3d...but we are getting ahead of ourselves!)**

Qualitative Features of the Anderson Model

- **The model ($w_i \in [-W, W]$ uniformly, μ given)**

$$-t\sum_{ij}\left(c_i^\dagger c_j + \textbf{h. c.}\right) + \sum_i (w_i - \mu)\, n_i$$

- **We ask: What is the diffusion constant D for this system?**
- **We "feel": For a given μ when W/t is "small", D will be nonzero, but for W/t large, we should get $D = 0$...Anderson insulator!**
- **Key ideas**
  - **Lifshitz tails in density of states**
  - **Localized and extended states**
  - **Concept of the** *mobility edge*

Density of States of the Anderson Model

- Of course, we are thinking of the disorder averaged DOS (N number of sites)

$$g(\varepsilon) = \langle \frac{1}{N} \sum_{\alpha} \delta(\varepsilon - \epsilon_{\alpha}) \rangle$$

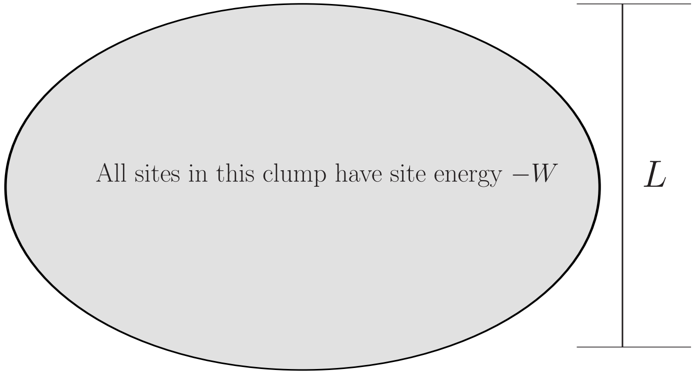
α runs over the (disorder dependent) one particle states

- First question: What is the "bandwidth" ("length" of energy range over which $g(\varepsilon) \neq 0$) as a function of in a "cubic" d dimensional lattice (when $W = 0$, i.e., for $g^0(\varepsilon)$ this is, of course, $4dt$)
- It is evident that the lowest possible energy eigenvalue is $-2dt - W$, and the highest is $2dt + W$ Question: How is this "evident"?... the bandwidth is $2(2dt + W)$

Density of States of the Anderson Model

- **The band bottom (top) are given by the energies** $\varepsilon^{\mp} = \mp(2dt + W)$
- **Now the question is what is the form of the density of states near the band bottom, i. e., near** $\varepsilon^{-} = -(2dt + W)$
- **This is most easily answered in the case of the "alloy disorder", i. e., in the Anderson model where** w **takes on one of two values** $-W$ **or** W **with probability** $(1 - x)$ **and** x **respectively (think of the TiAl problem)**

Density of States of the Anderson Model

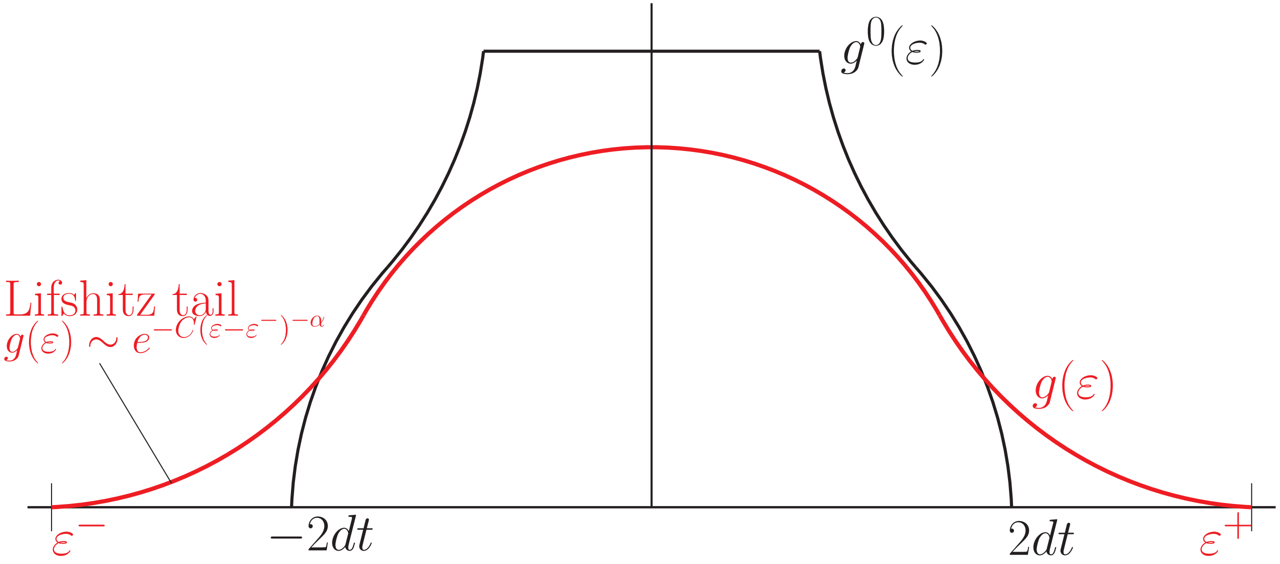


- **Since w is random (uncorrelated from site to site), there could be isolated regions of volume $\sim L^d$–"clump"– where all sites have energy $-W$...the probability of this is $P(L) \sim (1-x)^{L^d} = e^{-CL^d}$, $C = |\ln(1-x)|$**
- **The lowest "band state" in such a clump will have an energy near $-dt - W$...in fact, the energy of the lowest state will be proportional to $\varepsilon_c = \varepsilon^- + \frac{A}{L^2}$ where the constant A depends on the shape of the clump**
- **Thus, for $\varepsilon_c \approx \varepsilon^-$**

$$g(\varepsilon_c) \sim P(L) \Longrightarrow g(\varepsilon_c) \sim e^{-C(\varepsilon_c - \varepsilon^-)^{-d/2}}$$

...we see that the dos gets "exponential tails" near the band bottom (and top)!

Lifshitz Tails



- **It turns out that this is a general feature of the disorder problem...the DOS ends up with** *exponential* **tails (called Lifshitz tails...note it is** *not* **Lifshiftz's tails!!) of the form** $g(\varepsilon) \sim e^{-C|\varepsilon-\varepsilon^{\pm}|^{-\alpha}}$ **where** α **is a positive exponent**
- **The Lifshitz exponent** α **depends, in general, on the disorder distribution, the spatial dimension** d **etc...so does the positive constant** C**...**
- **Note that since** $\int_{-\infty}^{\infty} \mathrm{d}\varepsilon\, g(\varepsilon) = 1$ **(Question: Why?), we see that there will be a** *reduction* **(compared to** $g^0(\varepsilon)$**) of the density of states near the band centre**

Nature of Wave Functions

- **Suppose we are in $d \geq 2$, then we know that an attractive potential can have a *bound state* only if it is “strong enough”**
- **Suppose I have disorder in the system...then there will be “small clusters” where there is effectively a strong attractive potential..**
- **We might expect some of the wave functions to be “localized”...**
- **What general things can we say about extended and localized wave functions? In particular, when (for what disorder parameter etc.) does the wave function become localized?**

Some Preliminaries

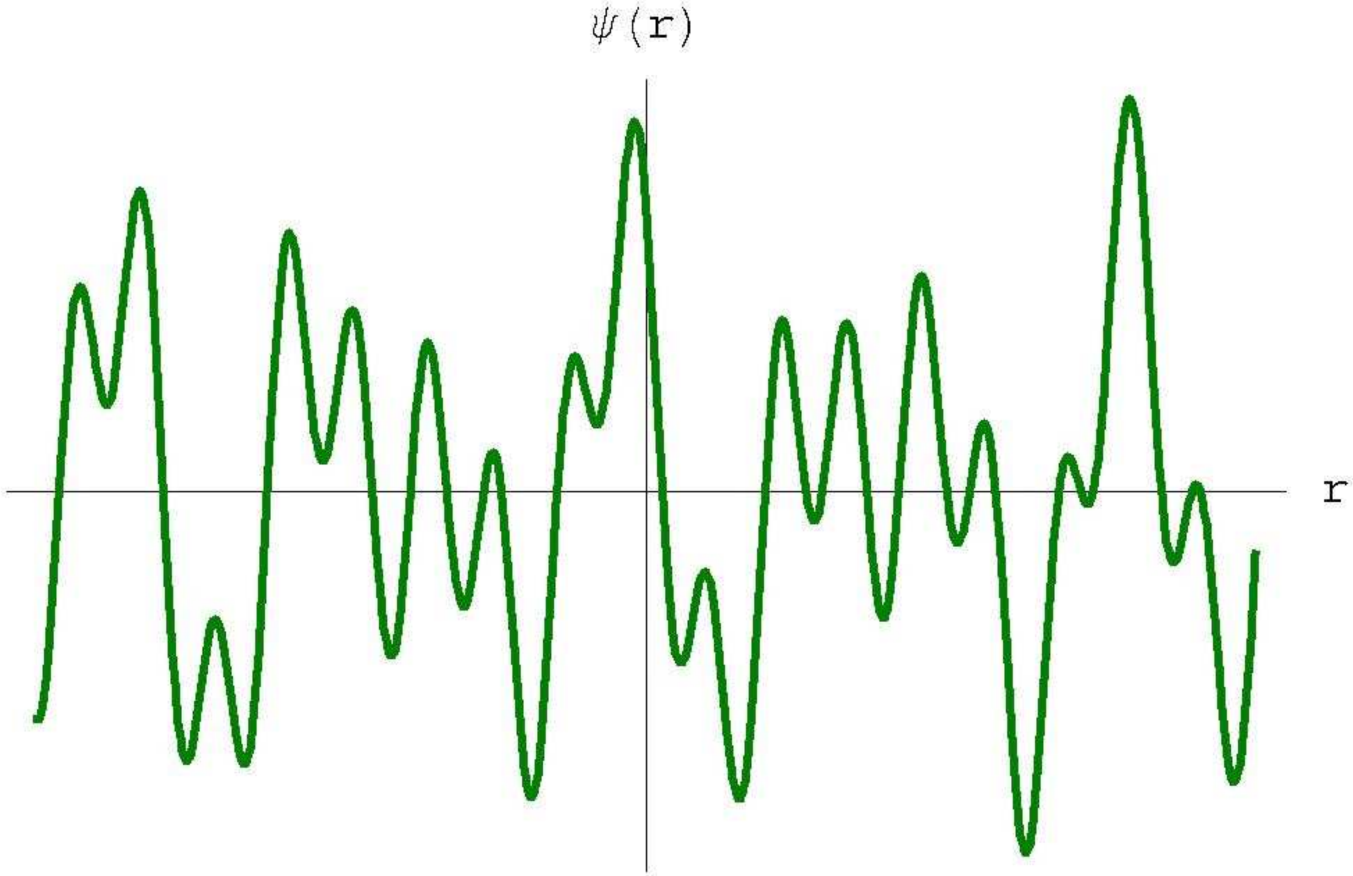
- **We recall some elementary things about states for further discussion**
- **The state $|j\rangle$ represents the one particle state at site j**
- **The one particle states of the disordered Hamiltonian are denoted by $|a\rangle$ (yes, $|a\rangle$ depends on the realization of the disorder)**
- **Clearly**

$$|a\rangle = \sum_j \langle a|j\rangle \; |j\rangle, \quad \textbf{with} \quad \sum j |\langle a|j\rangle|^2 = 1$$

- **By unitarity**

$$|j\rangle = \sum_a \langle j|a\rangle \; |a\rangle$$

Extended States



- **Extended states are "close to bloch states" and are "non zero" throughout the lattice**

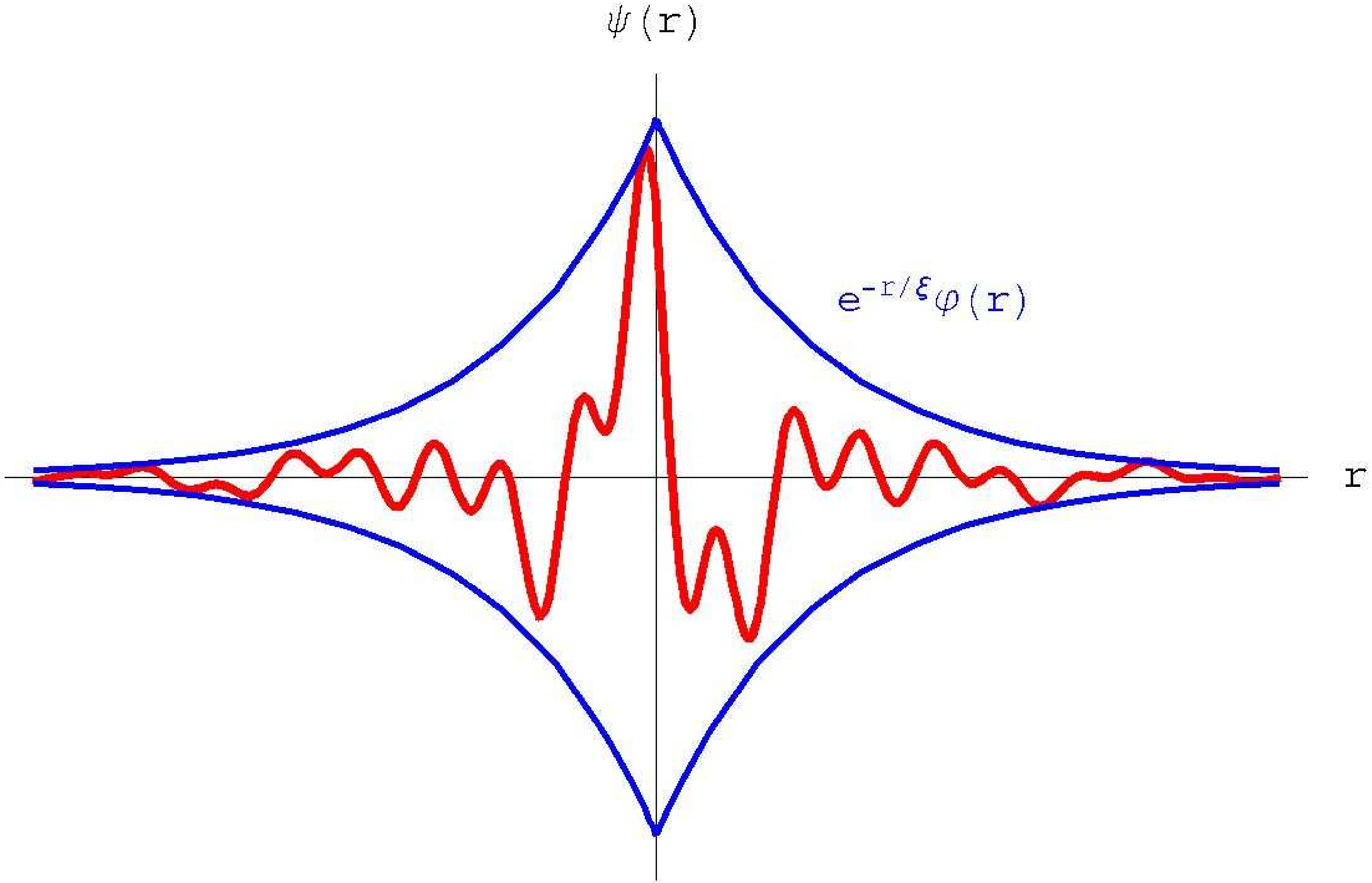
Extended States

- **A electron in such an extended state can** *carry current*
- **In such an extended state $|a\rangle$, very roughly**

$$|\langle a|j\rangle| \sim \frac{1}{\sqrt{N}}$$

where N is the number of sites in the system... This is the statement of the idea the wavefunction is nonzero throughout the lattice **Caveat: There are wave functions that satisfy this criterion, but *do not* carry currents...do you know any?**

Localized States



- **Analogous to bound states**

Localized States

- **Associated length scale ξ – called the *localization* length**
- **Does not carry a current**
- **If we calculate the projection of such a state on to particular sites, we will find that there are some j for which**

$$|\langle a|j\rangle| \sim 1$$

...indeed these are those sites around which the state is localized!

“Local” Questions

- **How can we tell if a given state is extended or localized?**
- **What determines which states are extended and which states are localized?**
- **Do we need a critical disorder to have localized states?**
- **...**
- **What has this got to do with diffusion?**
- **Our discussion (unless otherwise stated) will be valid only for 3d...2d will be discussed later in greater detail**

Deciding Between Extended vs Localized

- **A popular method is to calculate the Inverse Participation Ratio (IPR) for the given state defined as**

$$IPR(a) = \sum_j |\langle a|j\rangle|^4$$

- **For a Bloch state, IPR will turn out to be $\frac{1}{N}$, i. e., small for large N**
- **For a state fully localized at one site say $|i\rangle$, IPR will be of the order unity**
- **Why the fourth power? There is a deeper reason...related to diffusion!!**

Diffusion Again!

- **Ask the following question: Put an electron at site j at time $t = 0$, what is the probability $P_j(T)$ of finding the electron at this same site at time T in the limit $T \longrightarrow \infty$?**
- **Ask a $second$ question: Why ask the first question?**
- **Answer to the second question: If we find that $P_j(T)$ goes to zero, then we know that the electron is random walking...more interestingly, if we find that $P_j(T)$ is finite, then we know that the electron is not random walking!! If the second possibility is what we find, then we know that we will do not have diffusion!**
- **How do we find $P_j(T)$?**

Diffusion Again!

- **Finding $P_j(T)$ is quiet straightforward...let $|\psi(t)\rangle$ be the state of the electron at time t**
- **Clearly, $\psi(0) = |j\rangle = \sum_a \langle j|a\rangle \, |a\rangle$**
- **Now, since we "know" the one particle energy levels ϵ_a, we have**

$$|\psi(t)\rangle = \sum_a \langle j|a\rangle \, e^{-i\epsilon_a t} \, |a\rangle$$

- **The probability that the electron is in $|j\rangle$ at time t is**

$$\begin{aligned} P_j(t) = |\langle j|\psi(t)\rangle|^2 &= \left(\sum_a (\langle j|a\rangle)^2 \, e^{-i\epsilon_a t} \right) \left(\sum_b (\langle b|j\rangle)^2 \, e^{i\epsilon_b t} \right) \\ &= \sum_a |\langle j|a\rangle|^4 + \sum_{a \neq b} (\langle j|a\rangle)^2 (\langle b|j\rangle)^2 e^{-i(\epsilon_a - \epsilon_b)t}) \end{aligned}$$

Diffusion Again!

- To obtain $P_j(T), (T \longrightarrow \infty)$, one can argue that the *second term* does not contribute at large times, (Question: Argue this out! Suggestion: life can be made simple if you assume that ϵ_a are non degenerate) and we obtain

$$P_j(T) = \sum_a |\langle j|a\rangle|^4$$

- Now assume that all states are extended...then we see immediately that $P_j(T) \sim \frac{1}{N}$ and in the thermodynamic limit the particle diffuse away from $|j\rangle$!
- In the second scenario, assume that there is a localized state $|\ell_j\rangle$ "centered" around j...then we know that $|\langle \ell_j|j\rangle| \sim 1$...we thus see immediately that $P_j(T)$ is finite and independent of N! This means that the particle is *not* diffusing!
- We also see the connection between the "fourth power" and diffusion!
- We have now answered our first question, of how to tell between localized and extended states...we now move on the next question...

Which States are Localized?

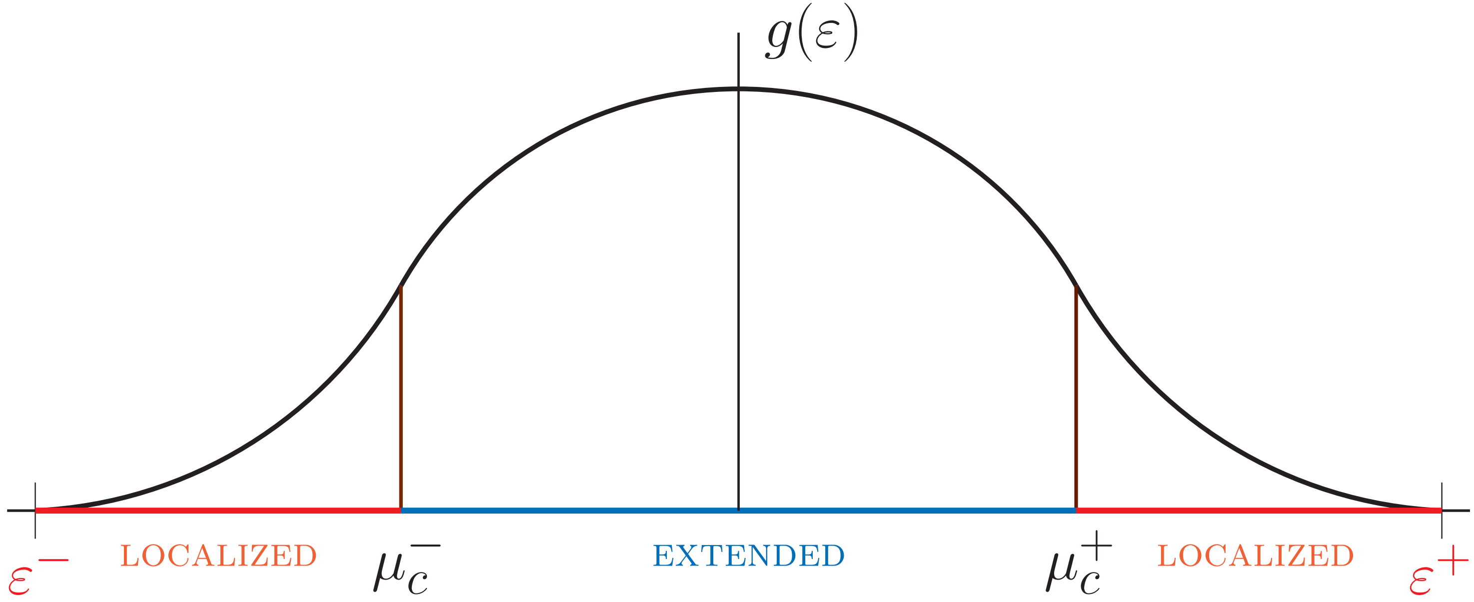
- **Consider an Anderson model with N lattice points in $3d$**
- **First question: For a given W, are there *any* localized states?**
- **Second question: If there are localized states, *"which"* states are localized?**
- **The answer the first question for the *Anderson model* is: There are localized states for *any* finite W!**
- **A moment's reflection will tell us that states very close to ε^-, i.e., states deep in the Lifshitz tail are localized...in fact we *used* this fact to show that there *are* Lifshitz tails!**
- **Conjecture: states at the band centre are extended?**

Which States are Localized?

- **Will there be extended states in the Lifshitz tails? Are there localized states in the centre of the band? Is there "coexistance"?**
- **More generally, we can ask given an energy ε, what fraction of the total states $g(\varepsilon)\mathrm{d}\varepsilon$ are *localized*?**
- **Mott provided the answer : At a given energy ε, all states (in 3d) are either localized or extended! There is no "coexistance"!**
- **Mott's Argument: Suppose for a given realization of disorder, there is a localized state coexisting with extended states (all of which are infinitesimally close to energy ε)...now for another realization of disorder which is *"infinitesimally" different* from the one above, the localized states will *hybridize* with the extended states and become extended!!**

The Mobility Edge

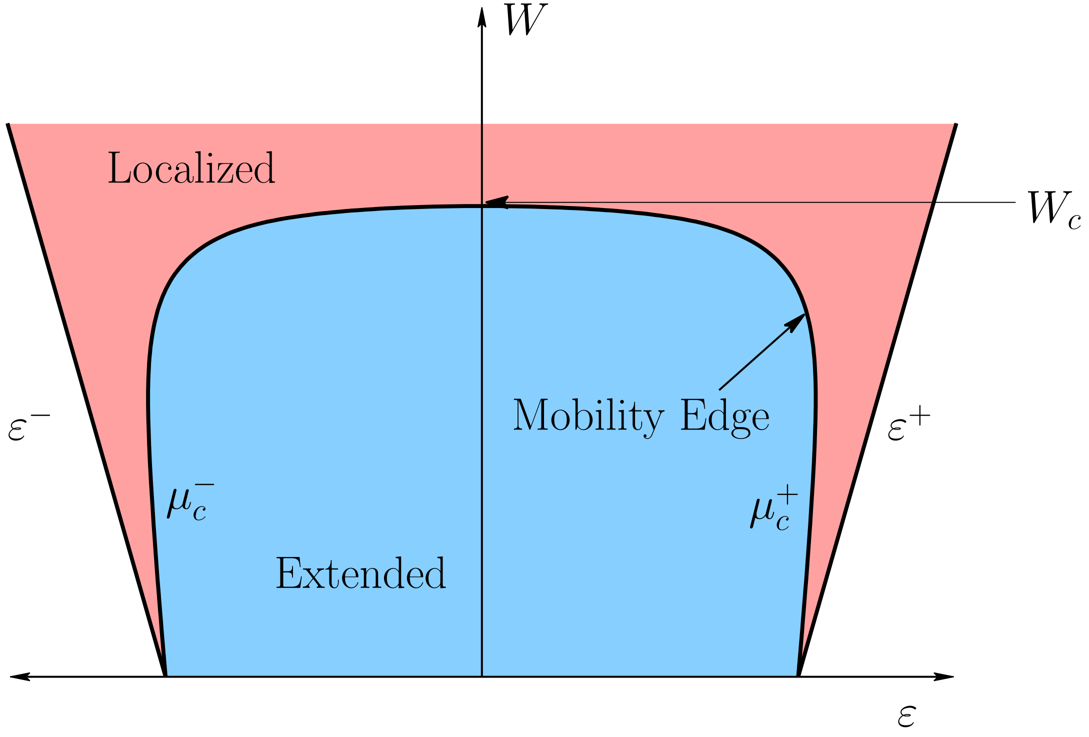
- **The arguments there exists an energy μ_c^- which depends on the disorder W for which all states are localized... similarly, there is a μ_c^+!**



- **The energy μ_c^- (μ_c^+) is called** *mobility edge* **Caveat: Mobility edges exist only in $3d$**
- **Natural question: How do $\mu_c^\pm$ evolve with W?**

How Mobility Edge Evolves with Disorder

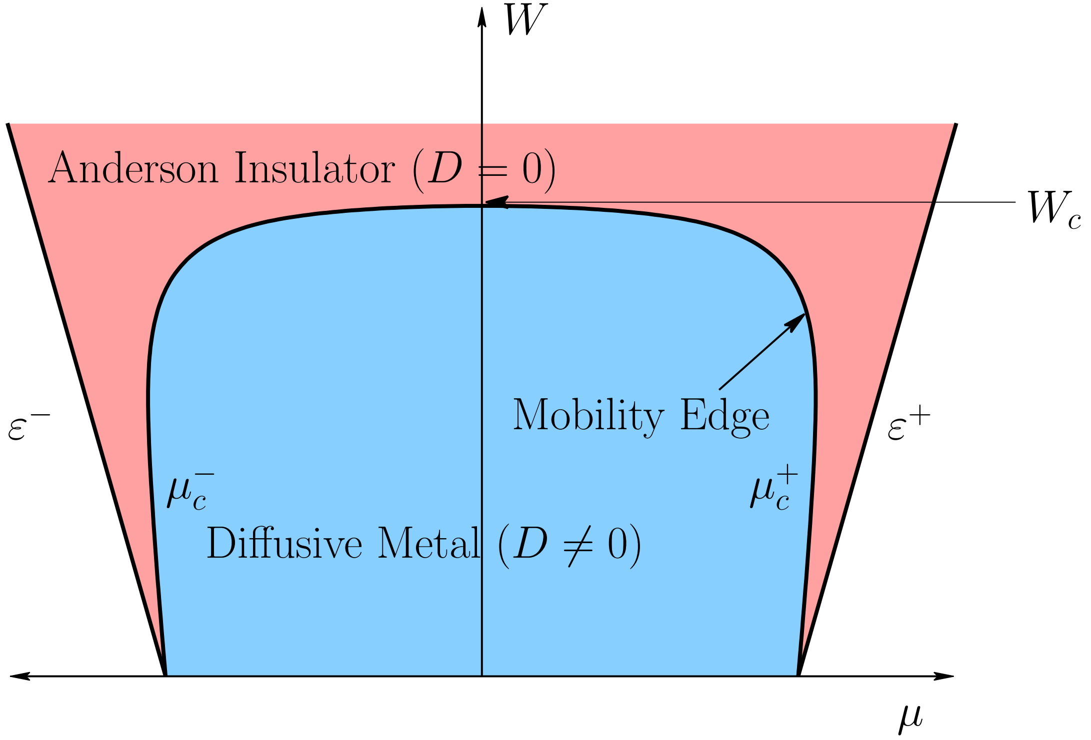
- **At large enough disorder we can "intuite" that the mobility edges will move towards the band centre**



- **In fact, at a critical W_c,** *all states* **become localized...it is this that was shown by Anderson in 1958**

The $3d$ Disorder Solution!

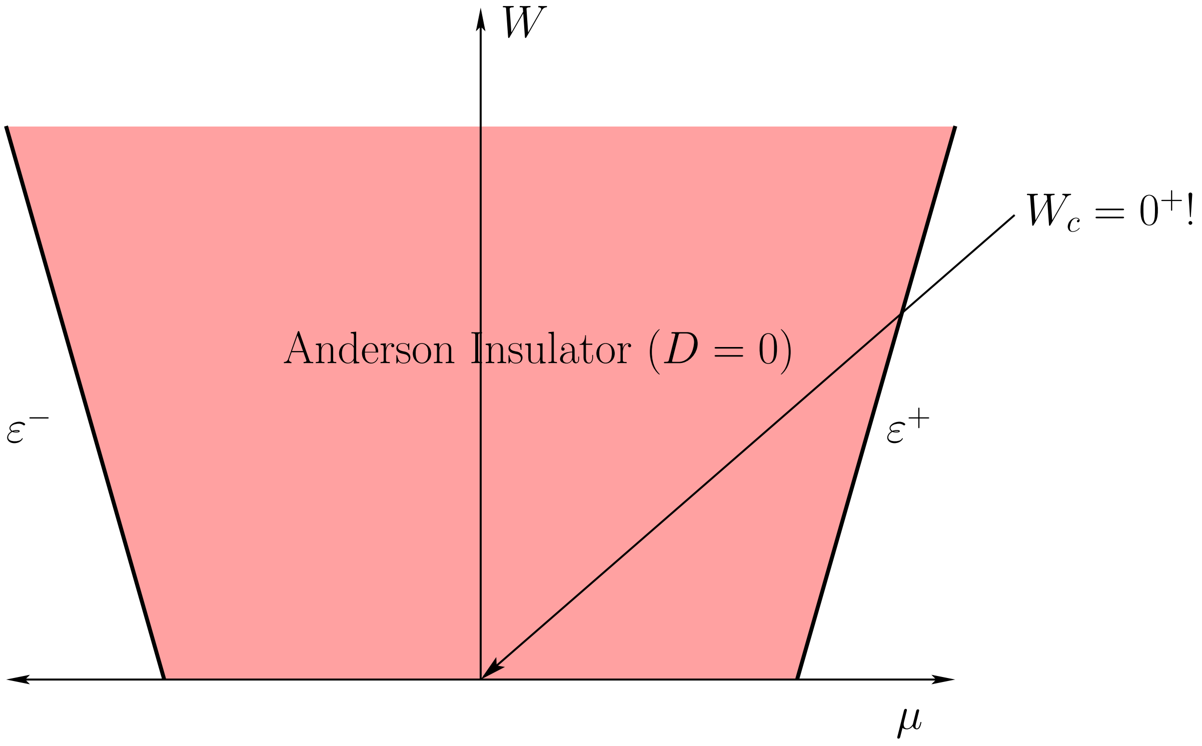
- **At a given level of disorder $W \leq W_c$, if the chemical potential μ lies in the extended states, i.e., $\mu_c^- < \mu < \mu_c^-$ we have a diffusive metal, else an Anderson insulator!**



- **This is the story in $3d$...It is quite different in $1d$ and $2d$!**

The $1d$ Disorder Solution!

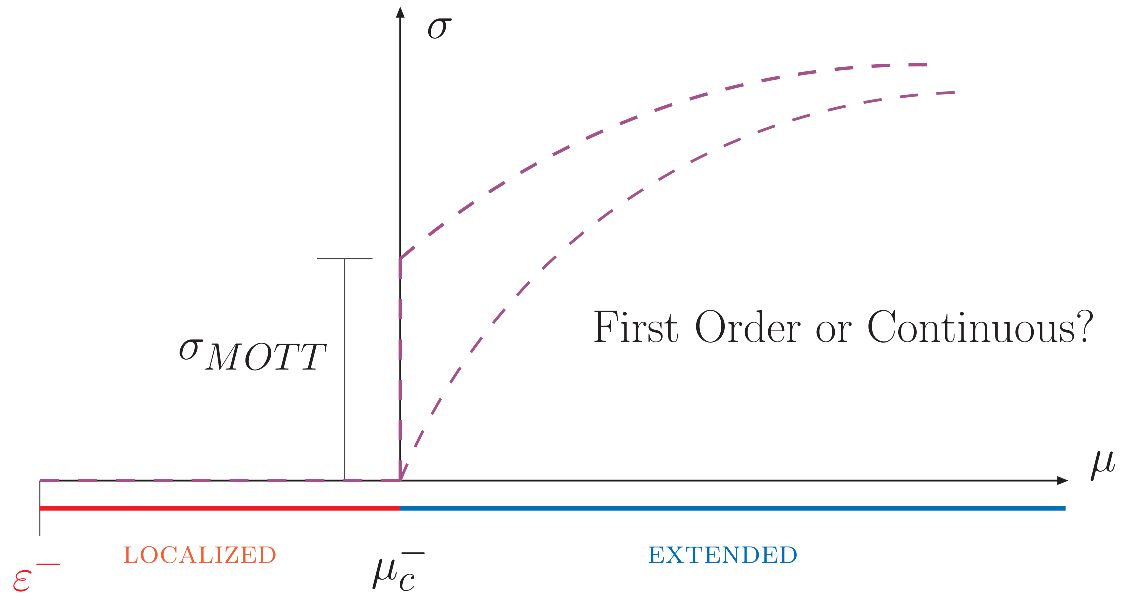
- **In $1d$ it turns out that W_c is zero (Mott and Twose)!** ***Any*** **amount of disorder will localize all states! There is "not enough room" in $1d$!**



- **$2d$ is a bigger story! We will see that $2d$ is the** ***"marginal"*** **dimension!**

The Conductivity Question

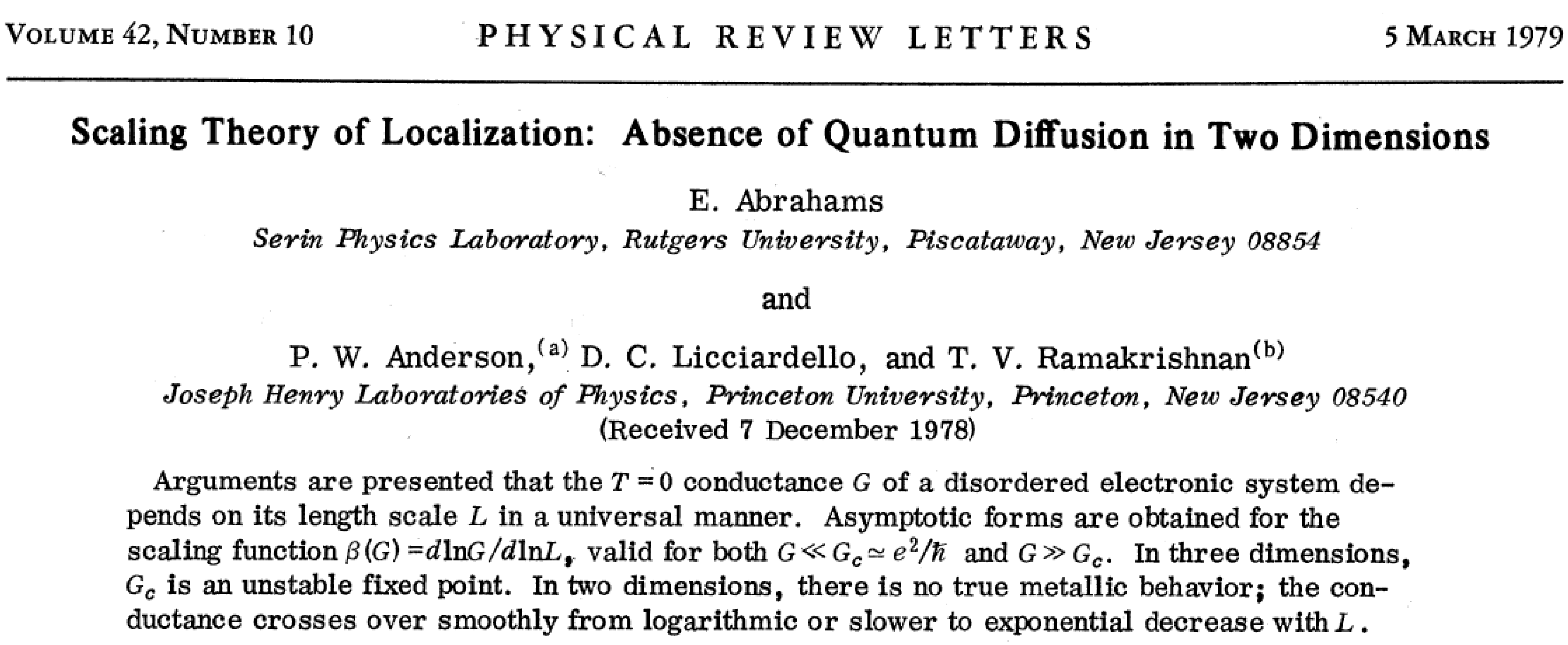
- **Back to 3d...A key question remains...what is the nature of the transition from a diffusive metal to an Anderson insulator?**



- **Is it a "first order transition" or a continuous one? Is there a minimum metallic conductivity?**
- **What about localized states? How does the localization length ξ change across the transition?**

The Scaling Theory

- **The Gang-of-Four Paper: takes off from the work of Thouless**



VOLUME 42, NUMBER 10 PHYSICAL REVIEW LETTERS 5 MARCH 1979

Scaling Theory of Localization: Absence of Quantum Diffusion in Two Dimensions

E. Abrahams
Serin Physics Laboratory, Rutgers University, Piscataway, New Jersey 08854

and

P. W. Anderson,(a) D. C. Licciardello, and T. V. Ramakrishnan(b)
Joseph Henry Laboratories of Physics, Princeton University, Princeton, New Jersey 08540

(Received 7 December 1978)

Arguments are presented that the $T=0$ conductance G of a disordered electronic system depends on its length scale L in a universal manner. Asymptotic forms are obtained for the scaling function $\beta(G) = d\ln G/d\ln L$, valid for both $G \ll G_c \simeq e^2/\hbar$ and $G \gg G_c$. In three dimensions, G_c is an unstable fixed point. In two dimensions, there is no true metallic behavior; the conductance crosses over smoothly from logarithmic or slower to exponential decrease with L.

- **Note "Arguments are presented..."!!**
- **Key points: Continuous transition in $3d$, no Mott minimum, no metal in $2d$!!**

Back to Thouless

- **Thouless (and friends) considered the following scenario:**
  - **Take an Anderson model at zero temperature – cube of size L, find its energy eigenvalues and states – "our cube"**
  - **Now think of making an Anderson model of size $2L$, by "connecting other cubes" to "our cube"**
  - **This is like adding a "perturbation to the boundary"...much like "twisting of the order parameter" to look for a stiffness...**
  - **How do we "model" the "effect" of the "other cubes" on states our original cube of size L?**
  - **...the story has two parts...**

The Thouless Story: Part 1

- **The effect of "other cubes" is to** *give a lifetime* **to the levels in our cube...**
- **The key physical idea is this: If the states are localized, then they** *will not* **be affected by boundary conditions...("paramagnet"), while they will "broaden" if delocalized (large L Question: Compared to what?)**
- **Let the broadening of the states (obviously those near the chemical potential which is kept fixed as the cubes are being patched) be $\Gamma(L)$ (this is an energy scale)**
- **Let the** *level spacing* **be $\Delta E(L)$**
- **The Thouless ratio $\mathcal{T}(L) = \dfrac{\Gamma(L)}{\Delta E(L)} = \mathfrak{g}(L)$, $\mathfrak{g}(L)$ is the dimensionless** *conductance*...**note** *conductance*!

The Thouless Story: Part 1

- **Relate $\Gamma(L)$ to diffusion $D(L)$ and call Einstein (G – conductance)**

$$\begin{aligned}\Gamma(L) = \frac{\hbar}{t_b(L)} = \frac{\hbar D(L)}{L^2} = \frac{\hbar\sigma(L)}{e^2 g_L(\mu) L^2} = \frac{\hbar\sigma(L) L^d \Delta E}{e^2 L^2} &= \frac{\hbar}{e^2} G(L)\,\Delta E(L) \\ \Longrightarrow \mathcal{T}(L) = \frac{\Gamma(L)}{\Delta E(L)} &= \mathfrak{g}(L)\end{aligned}$$

- **This is really an amazing result! If the states are fully localized, then $\Gamma(L)$ will get smaller and smaller as L goes to infinity compared to the level spacing $\Delta E(L)$...**
- **The key idea is that the behaviour of $\mathcal{T}(L) \equiv \mathfrak{g}(L)$ contains all the information about the localization problem...**
- **Now on to the second part of the story...**

The Thouless Story: Part 2

- **We need to know the behaviour of $\mathcal{T}(L)$ as we make L larger...we can ask this differently, what is $\mathcal{T}(2L)$?**
- **One might think that this depends on the details of the disorder etc...**
- **Thouless, based on numerical work and intuition, argued that $\mathcal{T}(sL)$ is determined only by $\mathcal{T}(L)$...**
- **The physical idea: if $\mathcal{T}(L)$ is large (states of L system are extended), then the state of the $2L$ system will be a strong admixture of states of the L systems, and will also be extended...thus $\mathcal{T}(L)$ appears to be sole quantity that controls the nature of the states as we scale the system...**
- **Stated as an equation $\mathcal{T}(sL) = f(s, \mathcal{T}(L))$!...Enter gang of four (Go4)!**

The Thouless Story: A bit more!

- **There is some more very useful information that we can learn from the Thouless picture..**
- **Suppose $\mathcal{T}(L)$ is "small", i. e., we have localized states at μ, then the wave functions will not further broaden when we have perturbations on the boundary**
- **This means that two localized states of that are close in energy** *will be separated far apart* **in space...why? A boundary perturbation will tend to mix states of same energy...thus if we know $\mathcal{T}(L)$ is small, this "mixing matrix element" must be small...this can happen only if the states are "far apart" , i. e., $O(L)$, thus "mixing matrix element" $\sim e^{-L/\xi}$**
- **We now see that $\mathfrak{g}(L) \sim e^{-L/\xi}$**

Go4 Scaling Theory

- **Based on the Thouless picture, Go4 write down the scale dependence of the dimensionless conductance $\mathfrak{g}$**

$$\frac{\mathrm{d}\ln \mathfrak{g}}{\mathrm{d}\ln L} = \beta(\mathfrak{g})$$

where $\beta(\mathfrak{g})$ is a *dimension dependent* **“beta-function”**

- **The key point is that the asymptotic forms of $\beta(\mathfrak{g})$ are universal and not determined by the nature and details of the disorder**
- **How do we obtain a functional form for $\beta(\mathfrak{g})$?**
- **This requires some physical input...**

Go4 Scaling Theory

- **Let us work in the large L limit and ask the nature of $\mathfrak{g}$ for "small" and "large" values...what is small and large?**
- **The dimensionless conductance is a "measure of the number of channels" available for electon flow...**
- **A bulk sample of a good metal will have a large value of $\mathfrak{g}$ – many many channels**
- **On the other hand a localized metal will have a very small $\mathfrak{g}$...**
- **Thus $\mathfrak{g}_0 \sim O(1)$ is the comparison point, $\mathfrak{g} \gg \mathfrak{g}_0 \Longrightarrow$ "bulk metal like" and $\mathfrak{g} \ll \mathfrak{g}_0$ "localized"**

Go4 Scaling Theory

- When $\mathfrak{g} \gg \mathfrak{g}_0 \implies$ **"bulk metal like", we get**

$$\mathfrak{g} \sim \sigma L^{d-2}$$

where σ **is the** *bulk conductivity* **of the metal which** *does not depend* **on** L**!**

- **It is immediate that**

$$\beta(\mathfrak{g}) = d - 2, \qquad \mathfrak{g} \gg \mathfrak{g}_0$$

- **It was shown that the** *quantum corrections* **(an correction to the Drudé formula) can be writtenas**

$$\beta(\mathfrak{g}) = d - 2 - \frac{a}{\mathfrak{g}}, \qquad \mathfrak{g} > \mathfrak{g}_0$$

a **is a constant**

Go4 Scaling Theory

- **When $\mathfrak{g} \ll \mathfrak{g}_0 \Longrightarrow$ "localized" regime,**

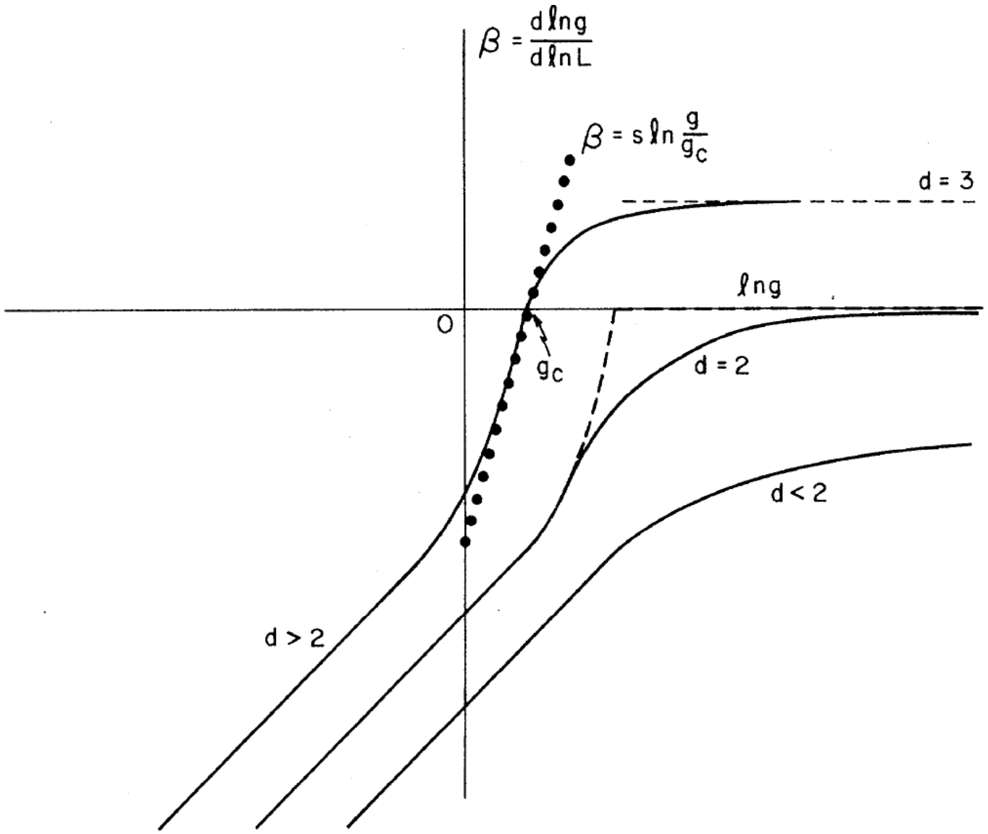
$$\mathfrak{g} = \mathfrak{g}_0 e^{-L/\xi}$$

- **Again, it is immediate that**

$$\beta(\mathfrak{g}) = \ln\left(\frac{\mathfrak{g}}{\mathfrak{g}_0}\right)$$

- **Lets see what this means...**

Go4 Scaling Theory



Gang of Four, 1979

- **Note the "nature is not unreasonable" assumption!**
- **We will investigate this in $d = 3$ first and then $d = 2$**

Go4 Scaling Theory $d = 3$

- **It is clear that $\beta(\mathfrak{g})$ changes sign at some value $\mathfrak{g}_c$!**
- **At this value of $\mathfrak{g}_c$, the dimensionless conductance** *does not change* **on changing L...this is a** *fixed point* **of the flow...**
- **Clearly, this is an** *unstable* **fixed point**
- **If $\mathfrak{g} > \mathfrak{g}_c$, the system flows to $\mathfrak{g} \to \infty$ as $L \to \infty$, i. e., it flows to a "Diffusive Fixed Point" (Note that $\mathfrak{g} = \infty$, is also a fixed point and is stable...now you see why we used the word DFP!!)**
- **If $\mathfrak{g} < g_c$, then flow is to $\mathfrak{g} \to 0$, i. e., to the Anderson Fixed Point!**
- **Finally! But what about Mott minimum etc?**

Go4 Scaling Theory $d = 3$

- **To ask about Mott, consider the case when $L \sim \ell$ (mean free path)**
- **Clearly $\mathfrak{g}_\ell$, the dimensionless conductance depends on the chemical potential μ of the system,i. e, we write $\mathfrak{g}_\ell(\mu)$**
- **Now if $\mu > \mu_c$ $(\mu_c \equiv \mu_c^-)$, then we expect $\mathfrak{g}_\ell(\mu) > \mathfrak{g}_c$, and $\mathfrak{g}_\ell(\mu_c) = \mathfrak{g}_c$**
- **What we want is the dependence of the $bulk$ conductivity $(L \to \infty)$ on $\mu - \mu_c$...for this let us record the following formula**

$$\mathfrak{g}_\ell(\mu) - \mathfrak{g}_c = (\partial_\mu \mathfrak{g})(\mu - \mu_c), \quad \mu - \mu_c \quad \text{“small”}$$

- **Strategy: Flow the $\mathfrak{g}$ to $L \to \infty$**

Go4 Scaling Theory $d = 3$

- **To solve the flow equation, note that near $\mathfrak{g}_c$**

$$\beta(\mathfrak{g}) = \frac{1}{\nu} \ln\left(\frac{\mathfrak{g}}{\mathfrak{g}_c}\right)$$

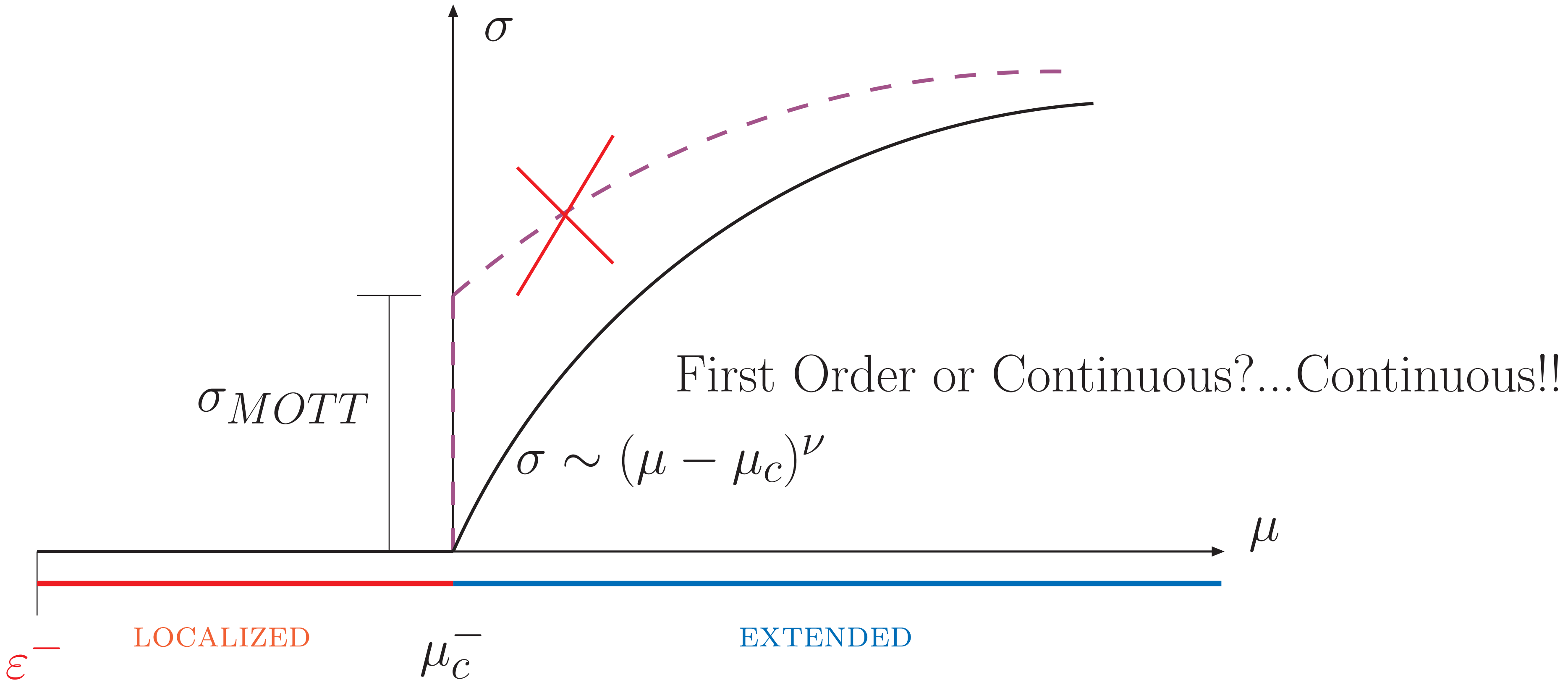
where $\frac{1}{\nu}$ is the slope of the $\beta vs \ln \mathfrak{g}$ curve at $\mathfrak{g}_c$ ($\nu > 0$)

- **A bit of algebra give**

$$\begin{aligned} \mathfrak{g}(L) &= B\mathfrak{g}_c \left[\frac{\mathfrak{g}_\ell - \mathfrak{g}_c}{\mathfrak{g}_c}\right]^\nu \frac{L}{\ell} = \sigma L \\ \Longrightarrow \sigma &= C(\mathfrak{g}_\ell - \mathfrak{g}_c)^\nu = C'(\mu - \mu_c)^\nu \end{aligned}$$

There is no minimum metallic conductivity!! Reason: quantum corrections!

Conductivity Answer $d = 3$



- **No metallic minimum! Mott is wrong – a rare occasion!**

Go4 Scaling Theory $d = 3$

- There is more! A harder look at the formula shows that, we can define a divergent length scal

$$\xi = \frac{\ell}{B}\left[\frac{\mathfrak{g}_\ell - \mathfrak{g}_c}{\mathfrak{g}_c}\right]^{-\nu}$$

such that

$$\sigma = \frac{\mathfrak{g}_c}{\xi}$$

- This is suggesting that this is a “critical phenomenon”
- What about the “other side”? $\mu < \mu_c$?

Go4 Scaling Theory $d = 3$

- **By a very similar calculation, we can work out what $\mathfrak{g}$ is doing for $\mu < \mu_c$...**

$$\mathfrak{g}(L) = \mathfrak{g}_c e^{-K\left|\frac{\mathfrak{g}_\ell - \mathfrak{g}_c}{\mathfrak{g}_c}\right|^\nu \frac{L}{\ell}} \Longrightarrow \xi = \frac{\ell}{K}\left|\frac{\mathfrak{g}_\ell - \mathfrak{g}_c}{\mathfrak{g}_c}\right|^{-\nu} \sim |\mu - \mu_c|^{-\nu}$$

- **This is beautiful! We get the same critical exponent ν as we got on the rightside!! Neat!**
- **What is ν?**

Go4 Scaling Theory $d = 3$

- **Use $d = 2 + \epsilon$**

$$\beta(\mathfrak{g}) = \epsilon - \frac{a}{\mathfrak{g}}, \qquad a = \epsilon \mathfrak{g}_c$$

- **With a bit of further algebra, we get**

$$\nu = \frac{1}{\epsilon}$$

- **In 3d, we get $\nu = 1$ (for small enough values of 1)...**
- **This is the story in 3d...what happens in 2d?**

Go4 Scaling Theory $d = 2$

- Without the quantum correction term $\beta(\mathfrak{g}) = 0$ (this is why $d = 2$ is called the marginal dimension)
- Suppose we start with $\mathfrak{g}_\ell$ (just as we did before), then we have (due to quantum corrections)

$$\beta(\mathfrak{g}) = -\frac{a}{\mathfrak{g}} \Longrightarrow \mathfrak{g}(L) = \mathfrak{g} - C \ln\left(\frac{L}{\ell}\right)$$

i. e., a large enough system becomes an insulator!

- We say that there is no metal in 2d!
- We will later see that is the result of "weak localization"...a purely quantum effect!

Summary

- **Thouless arguments, ratio $\mathcal{T}$ of broadening and level spacing is same as dimensionless conductance...depends on the scale L of the system, $\mathcal{T}(2L)$ depends only on $\mathcal{T}(L)$**
- **Go4 – scaling of $\mathfrak{g}$ depends only on $\mathfrak{g}$**
- **Forms of β function from simple arguments, key quantum correction term added**
- **Results ($d = 3$): No Mott minimum, diverging scales, critical exponents**
- **Results ($d = 2$) : Marginal dimension, quantum corrections play crucial role, no metal in the thermodynamic limit!**
- **Next we will understand what "quantum correction" means!**