- Time reversal in quantum Mechanics

$$\begin{array}{ccc} |\psi\rangle & \xrightarrow[t_0]{F(e^{-iHt_0})} & |\psi(t_0)\rangle \\ \Big\downarrow \Theta & & \Big\downarrow \Theta \\ & & \Theta|\psi(t_0)\rangle \\ & & \| \; ? \\ |\tilde\psi\rangle & \xrightarrow[B(e^{+iHt_0})]{} & |\tilde\psi(-t_0)\rangle \end{array}$$

(for all $\tilde\psi$)

If the equality is satisfied, then the system has TR symmetry.

$$\Rightarrow \quad \Theta e^{+iHt} = e^{-iHt}\Theta \quad \forall t$$

$\Rightarrow$ For consistency we ~~have~~ conclude

Θ is <u>antilinear</u>

$$[\Theta, H] = 0.$$

$\Rightarrow$ We ~~know~~ have $\Theta|r\rangle = |\vec r\rangle$ and $\Theta|\vec p\rangle = |-\vec p\rangle$

Also have operator ~~$\Theta^{-1} + \Theta \ldots$~~

$$\left.\begin{array}{c}\vec r\\ \vec p\\ \vec L\end{array}\right\} \xrightarrow{\Theta} \left\{\begin{array}{c}\vec r\\ -\vec p\\ -\vec L\end{array}\right.$$

What is the action of Θ on the spin Hilbert space?

expect $\vec S \xrightarrow{\Theta} -\vec S$ $\boxed{\text{(Ex : Work out commutation re}}$

If I have spin s $\quad |\psi\rangle = \sum_{m=-s}^{s} \psi_m |m\rangle$ (the spin st

Now we need a math theorm to proceed

$$\Theta = \mathcal{U} K$$

where $\mathcal{U}$ is unitary and K is the "complex conjugation operator". If $|a\rangle$ is a basis then $|\psi\rangle = \sum_a' \psi_a |a\rangle$

$$\Theta|\psi\rangle = \mathcal{U} K |\psi\rangle = \mathcal{U} K \sum_a' \psi_a |a\rangle$$

$$= \mathcal{U} \sum_a' \psi_a^* K|a\rangle \qquad \text{Take } K|a\rangle = |a\rangle \text{ (convention)}$$

$$= \quad \Theta|\psi\rangle = \sum_a' \psi^* |a\rangle \; \mathcal{U}|a\rangle.$$

Ex: Find $\mathcal{U}, K$ in $|r\rangle$ and $|p\rangle$ basis

Now apply this theorem to our spin S system

$$\mathcal{U} K \sum_m' \psi_m |m\rangle = \sum_m' \psi_m^* \, \mathcal{U}|m\rangle.$$

Now we expect $\mathcal{U}|m\rangle$ to be parallel $|-m\rangle$.

One can argue that

$$\mathcal{U} = e^{-i\pi S_y} \quad \text{will do the job!}$$

Let us see what happens for spin-half $S_y = \frac{1}{2}\sigma_y$

$$\mathcal{U} = e^{-i\frac{\pi}{2}\sigma_y} = -i\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Thus $\Theta|\uparrow\rangle = |\downarrow\rangle$ ~~and $\Theta|$~~

$\Theta|\downarrow\rangle = -|\uparrow\rangle$ or $\Theta|\sigma\rangle = \sigma|\bar{\sigma}\rangle$ $(\bar{\sigma} = -\sigma$

But $\Theta|\psi\rangle = \Theta\left(\sum_\sigma \psi_\sigma |\sigma\rangle\right)$

$= \sum_\sigma \sigma \psi_\sigma^* |\bar{\sigma}\rangle$!

$\Theta^2|\psi\rangle = \sum_\sigma \sigma \psi_\sigma^* \bar{\sigma} |\sigma\rangle$

$= \sum_\sigma \sigma\bar{\sigma} \psi_\sigma |\sigma\rangle = -|\psi\rangle$!

$\Theta^2 = -1$! This is true for all [$(2s+1) =$ ~~odd~~ even].
($\Theta^2 = 1$ [$(2s+1)$ is ~~even~~ odd].

So $\Theta|p\sigma\rangle = \bar{\sigma}|-p\bar{\sigma}\rangle$!

$\Theta^2 = -1$

Now we find

Claim: If ε is an energy eigenvalue of ~~$\mathcal{H}$~~ ~~1/2 integral spin $\mathcal{H}$ spin~~ TR invariant $\mathcal{H}$ ($\frac{1}{2}$ interal spin), then, the level is at least two-fold degenerate.

$\mathcal{H}|\psi\rangle = \varepsilon|\psi\rangle$.

Assume $|\psi\rangle$ is non degenerate. then
~~$\Theta|\psi\rangle$ is Since~~ Now $\Theta|\psi\rangle$ is also an eigenstate with the same energy ($[\Theta, \mathcal{H}] = 0$)

$\Theta|\psi\rangle = e^{i\alpha}|\psi\rangle \Rightarrow \Theta^2|\psi\rangle = e^{-i\alpha}\Theta|\psi\rangle$

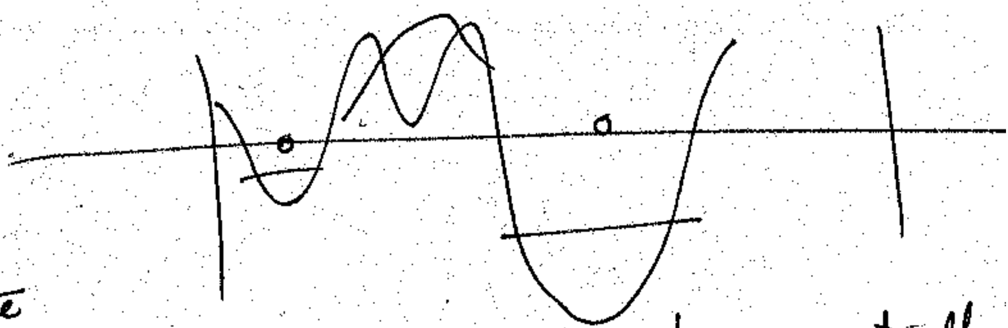
$= -|\psi\rangle = e^{-i\alpha}e^{i\alpha}|\psi\rangle =$

obtain a contradiction.

$\Theta|\psi\rangle$ and $|\psi\rangle$ are diffent states!

ε is atleast two fold degenerate.

TB models



Instead of the full Hilbert space $\{|\vec{r}\rangle|o\rangle\}$ go to a unitcell; pick "important" orbitals $|I a \sigma\rangle$ (I → unit cell, N; a → orbital)

We now write a hopping Hamiltonian

$$\mathcal{H} = \sum_{I}\sum_{\vec{\delta}} t_{ab}(\vec{\delta})\, |I+\vec{\delta}, a, \sigma\rangle\langle I b|$$

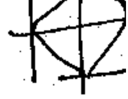
($\vec{\delta}$ are ~~n~~ nearest neighbor) lattice vectors)

One can again Bloch/Block diagonalize

$$\mathcal{H} = \sum_{k} H_{ab}(k)\, |k a \sigma\rangle\langle k b \sigma|$$

($H(k)$ is a $n\times n$ mat... get n band... $2n$ ba... Metals insulat...)

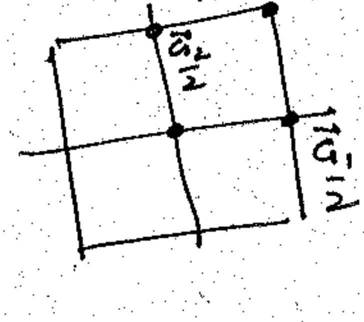
where $|\vec{k} a\sigma\rangle = \frac{1}{\sqrt{N}}\sum_{I} e^{i\vec{k}\cdot\vec{R}_I}|I a\sigma\rangle$

(Square lattice 

Keep spin aside for a second →

Take $|\vec{k}\rangle$ what is $\Theta|\vec{k}\rangle \overset{?}{=} |-\vec{k}\rangle$.

Now not that on a lattice very nice thing happen



$\Theta|\frac{\vec{G}_1}{2}\rangle = |-\frac{\vec{G}_1}{2}\rangle$

But $|-\frac{\vec{G}_1}{2}\rangle \equiv |\frac{\vec{G}_1}{2}\rangle$!

There are more TRIMs!

Example Graphene $\boxed{\text{Figure}}$

$$\cancel{H}\ \mathcal{H} = \sum_{a,b=A,B} \cancel{\mathcal{H}(k)}\ \cancel{\mathcal{H}(k)}\ H_k\, c^\dagger_{ka\sigma} c_{b\sigma}$$

$$H_k = \begin{pmatrix} 0 & f(k) \\ f^*(k) & 0 \end{pmatrix} \qquad f(k) = -t\left(1 + e^{-i\vec{k}\cdot\vec{a}_1} + e^{i\vec{k}\cdot\vec{a}_2}\right)$$

~~We~~ $E(k) = \pm |f(\vec{k})|$

~~One can show $f(\vec{k}) \neq 0$~~

We get two Bands, which touch each other at $\pm\vec{K}$. $f(\pm\vec{K}) = 0$.

Suppose $\vec{k} \to \vec{K} + \vec{k}$, near

$$f(\vec{k}) = v\,(k_x - ik_y) \qquad v = \hbar t a$$

$$\cancel{H_k}\ \mathcal{H}^{\vec{K}} = \vec{\sigma}\cdot\vec{k} \;\to\; \text{Dirac } (\text{W} \cancel{\text{Weyl}})\ (\cancel{\text{Weyl}}) \text{ particles}$$

I have one particle per site, each band is 2fold spin degenerate ($\mu = 0$), All states in bottom band are occupied.

There are gapless excitation $C_v \sim T^2$ $\boxed{\text{show this}}$

(not the usual metals, but metallic semi metal)

Now can we get an insulator starting from graphene?

Yes! (take $v=1$)

$t_{AA}(\vec{0}) = m$ and $t_{BB}(\vec{0}) = -m$

$$H_k = \begin{pmatrix} m & f(k) \\ f^*(k) & -m \end{pmatrix} \qquad E(k) = \pm\sqrt{m^2 + |f(k)|^2}$$

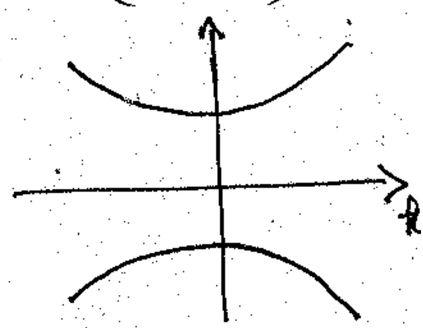
~~Okay~~, we ~~have~~ get an insulator ~~an insulator~~ with 1 particle per site ($\mu = 0$).

Near the ~~band~~ K

$$H_k = m\sigma_z + \sigma\cdot\vec{k}$$

$$\mathcal{E}(k) = \pm\sqrt{m^2 + |k|^2}$$

($v = 1$)



Now I ask the question

Is there a state with energy $\mathcal{E}(k)$ m

Need to solve for $\mathcal{E}(\vec{k}) = m$.

obvious answer $\vec{k} = 0$. But are there nonzero $\vec{k}$s ?

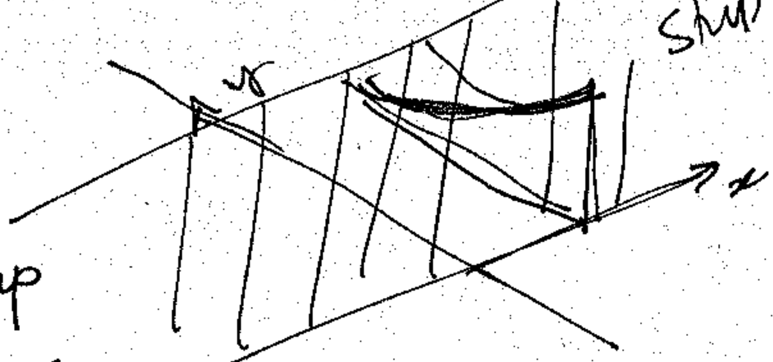
$$m = \sqrt{m^2 + k_x^2 + k_y^2}$$

Suddenly you realize it is possible if k_y is purely imaginary

Solution $= (k, -ik)$

Wavefunction $\psi \sim e^{ikx} e^{-ky}$

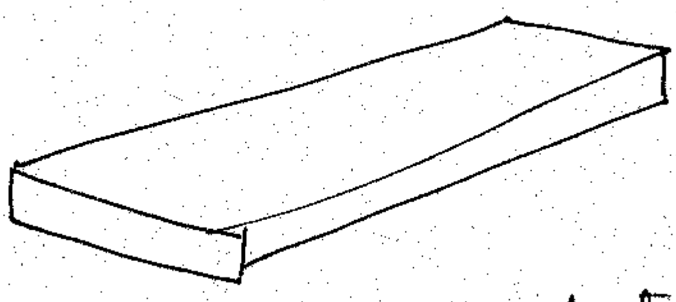
is not bounded as $y \to -\infty$. So

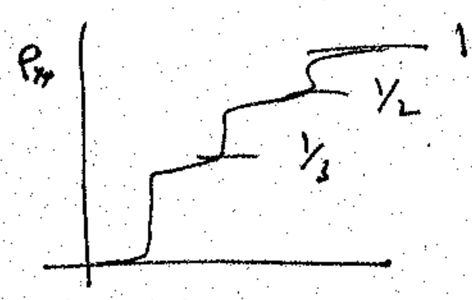


Suddenly, we see we can have midgap states. But they have to live on the edge!

Edge states are present ~~in bolts~~ in graphene (in fact they are peculiarly close to the chemical potential and indeed control the response of graphene ribbons).

Now we ask the question: Do we always need a lattice to obtain an insulator in a clean system?
The answer is an overwhelming no!
Enter: Quantum Hall Effect





Now we have a density ρ of electrons
ρ = Number/unit area

We have

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

Note in 2D σ has units of conductance.

Now, we must have $\sigma_{yx} = -\sigma_{xy}$ (Show this)

This means that if the system has TR symmetry (show this).

$\sigma_{xy} = \sigma_{yx} = 0$

Now $\cancel{j_x}$

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \frac{1}{\cancel{\sigma_{xx}\rho_{yy}}} \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix}$$

$$= \quad \underline{\sigma} = \begin{pmatrix} \rho_{yy} & -\rho_{xy} \\ \rho_{xy} & \rho_{xx} \end{pmatrix} = \frac{1}{\rho_e^2 - \rho_H^2} \begin{pmatrix} \rho_{yy} & -\rho_H \\ \rho_H & \rho_{yy} \end{pmatrix}$$

$\cancel{\sigma_2} =$ $\rho_e = 0$ $\quad \rho_H = \frac{h}{e^2 n}$ ← integer

$$\Rightarrow \sigma_{xy} = -\sigma_H = -\frac{e^2}{h} n \longrightarrow \text{integer!}$$

[The real puzzle is that $\sigma_H = \frac{h\rho}{eB} \frac{e^2}{h}$ $\left(\frac{\rho e}{B}\right)$

$= \left(\frac{h\rho}{eB}\right) \frac{e^2}{h}$ (exercise)

Should always be true by Lorentz (Galilean) invariance.]

$$\sigma_H = \frac{\rho e}{B} = \frac{\rho}{\frac{eB}{h}} \frac{e^2}{h}$$

$$= \underbrace{\rho/(B/\phi_0)}_{\nu} \frac{e^2}{h} = \nu \frac{e^2}{h}$$

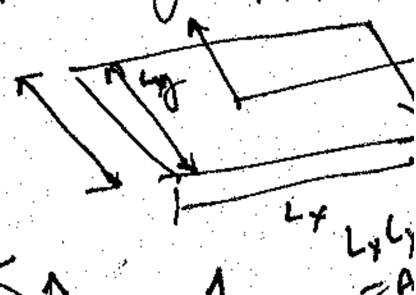
But instead of $\nu \frac{e^2}{h}$, we get steps.] .

Stay clear of this and stick to $\nu = n$..

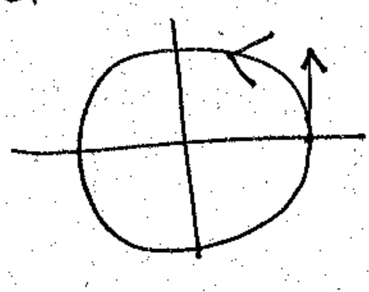
<u>okay</u> set up the problem

$$\vec{A} = (-By, 0, 0) \quad \text{(Landau gauge)}$$

$$\vec{B} = B e_z$$



Classical motion of the electron:



Identify some scales.

$$\omega_c = \frac{eB}{m}$$

$$\boxed{\cancel{\text{length scale} \ \frac{\hbar^2}{2m}}} =$$

$$H = \frac{\hbar^2}{2m} (\vec{p} + e\vec{A})^2 \quad \text{(keep spin aside)}$$

$$= \frac{1}{2m} \left(-i\hbar \frac{\partial}{\partial x} + eBy\right)^2 - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2}$$

- What are the symmetries $\rightarrow T_x$

No time reversal (Show).

Scales $\hbar \omega_c \rightarrow$ energy scale

$$\frac{\hbar^2}{m}\frac{1}{\ell^2} = \hbar \omega_c = \frac{e\cancel{B}}{\cancel{m}}$$

$$\frac{1}{\ell^2} = \frac{m\omega_c}{\hbar} = \frac{eB}{\hbar} = l_B =$$

$$\Rightarrow \quad 2\pi l_B^2 = \frac{h}{e}/B = \frac{\phi_0}{B}$$

$2\pi l_B^2$ - area of ~~the~~ ~~acte~~ circle enclosing unit flux.

~~The~~ Since T_x is a symmetry, we get

$$\psi(x,y) = e^{ik_x x}\phi(y)$$

Put in

$$\frac{\hbar^2}{2m}\left(\hbar k_x - \cancel{eB}\,eB\,y\right)^2 + \frac{p_y^2}{2m}$$

$$\frac{1}{2}m\omega_c^2\left[l_B^2 k_x - y\right]^2 + \frac{p_y^2}{2m}$$

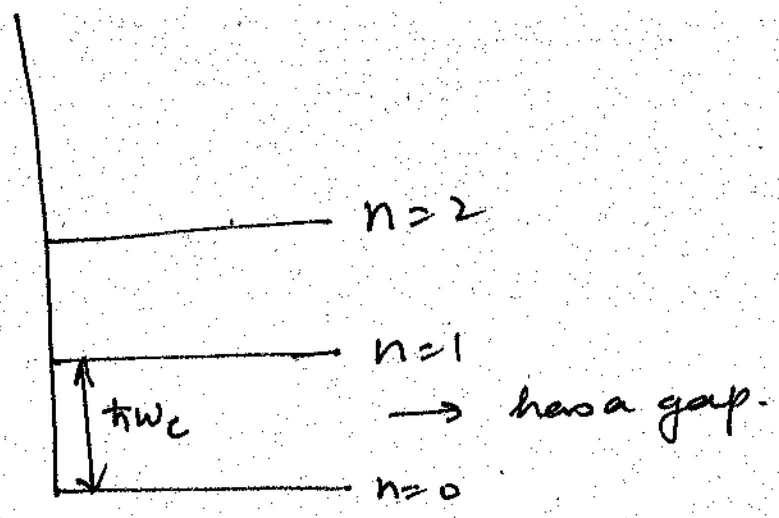
$$\frac{p_y^2}{2m} + \frac{1}{2}m\omega_c^2\left(y - l_B^2 k_x\right)^2$$

"Shifted Harmonic" oscillator

$$\cancel{\varepsilon_{nk_x}}\ \varepsilon_{k_x n} = (n+\tfrac{1}{2})\hbar\omega_c$$

$$\psi_{k_x n}(x,y) = e^{ik_x x}\phi_n^{SHO}(y - l_B^2 k_x).$$

Note all the states associated with k_x are localized around $d(k) = l_B^2 k_x$



What ~~is~~ is the degeneracy of a level?

Note $\quad d(k) = l_B^2 k_x \qquad -\frac{L_y}{2} \leq l_B^2 k_x \leq \frac{L_y}{2}$

$k_x = \frac{2\pi}{L_x} n$ for periodicity in the x direction

$D_{\text{geny}} = 2 n_{max} \qquad \frac{2\pi}{L_x} n_{max} = \frac{L_y}{2 l_B^2}$

$$\Rightarrow \quad \text{Degeneracy} = \frac{L_x L_y}{2\pi l_B^2} = \frac{A}{2\pi l_B^2}$$

$$= \frac{A}{\Phi_0/B} = \frac{B}{\Phi_0} A = \frac{\text{Total Flux}}{\text{flux quanta}}$$

Alternatate way to understand these result.

Filling factor $\nu = \frac{\rho A}{\text{Degn}} = \frac{\rho}{B/\Phi_0}$

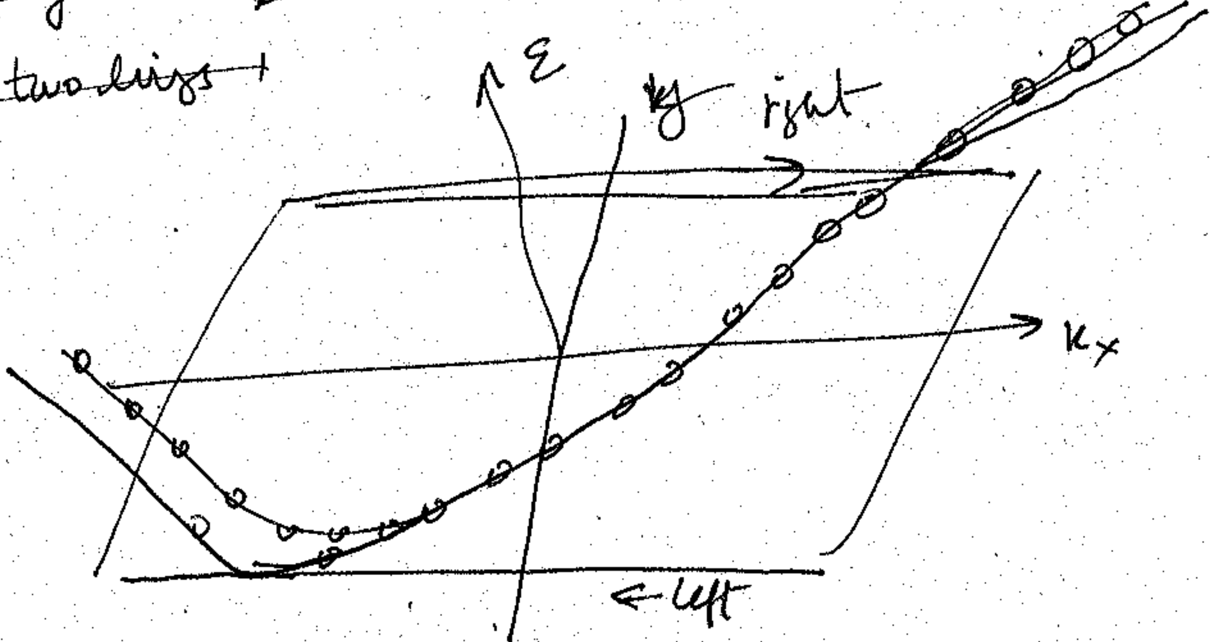
If ~~the~~ I have $\nu = 1, 2, 3, n$ I have fully filled Landau level and I have an insulator. There is a gap to excitation.

Now we ask: Does this ~~insulator have~~ ~~we should expect~~ system have edge states?

We need to make a finite strip of this. Add a box potential $V=\infty$ $|y| \geq \frac{L_y}{2}$.

McDonald & Streada, have shown that this system has edge states, whose energies are in the bulk gap!

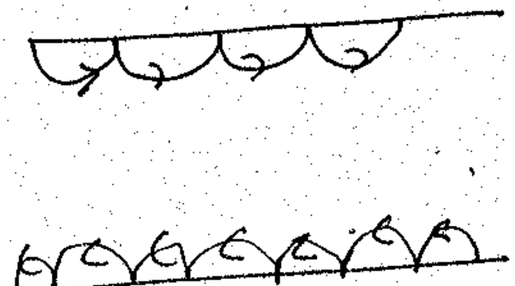
~~Note two things~~



① All edge states on the top edge are right movers.

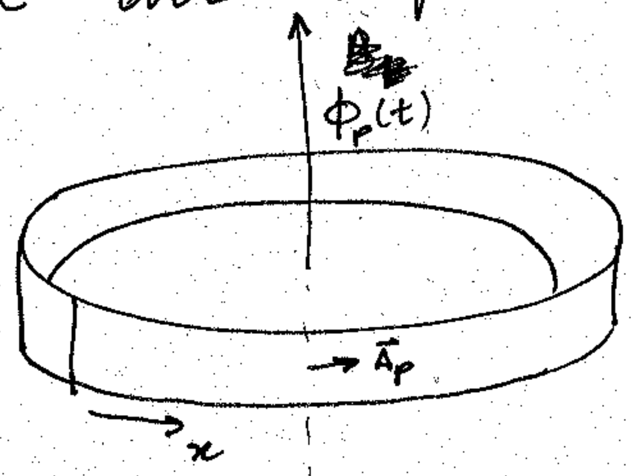
② ~~Even~~ There is a gap between the edge states of different Landau levels.

Edge states should exist



Now we are ready for Laughlin's flux insertion argument.

Let us take our slip and wrap it around

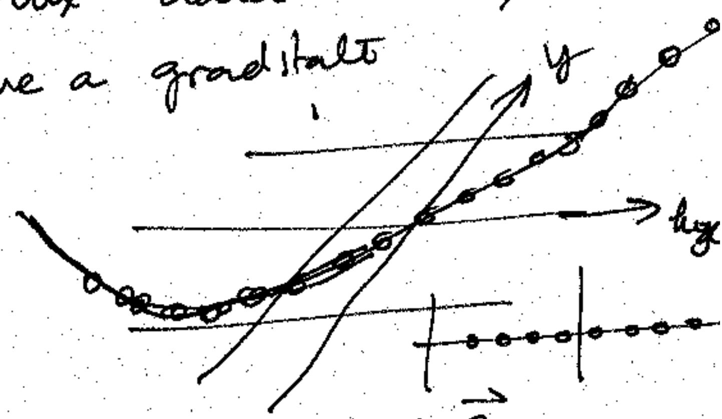


$$\oint \vec{A}_p \cdot dx = \phi_p(t)$$

The idea is to insert a flux adiabatically.

Before flux insertion, we have a ground state

statement is that



$$H = \frac{1}{2m}(\vec{p} + e\vec{A} + e\vec{A}_p(t)).$$

* $\vec{A}_p(t) = A_p(t)\,\vec{e}_x$

~~$\vec{A}_p(t)$ is a constant flux~~

The idea now is to say $\vec{J} = -\dfrac{\partial \langle H(t)\rangle}{\partial \vec{A}_p}$

or

$$J_x = -\frac{\partial \langle H(t)\rangle}{\partial A_p}$$

Now under the action of A_p which is constant

$\hbar k_x \to \hbar k_x + e A_p$.

Now what happens when $e A_p = \dfrac{2\pi}{L}\hbar$

But $A_p = \frac{h}{eL}$ $=$ $\int A_p \, dl = \frac{h}{e} = \Phi_0$!!

$\Delta A_p \sim \frac{\phi_0}{L}$

So restoring the system

$$J_x = -L \frac{\Delta U}{\phi_0}$$

$$j_x = -\frac{\Delta U}{\phi_0} \quad \text{by } \Delta U = eV_H$$

Since one state in the lower edge was taken out and an extra was put in on the top edge.
If here one n Ladau levels occupied,

$$\Delta U = neV_H$$

then

$$\frac{j_x}{V_H} = -\frac{ne}{\phi_0} = -\frac{ne^2}{h} = -\cancel{\sigma_H}$$

$$\boxed{\sigma_H = \frac{ne^2}{h} \;!}$$

We also see something quite remarkable. There The current is carried by the edge state. There are only forward moving edge state on the top edge so, ~~there~~ Since there are no backward moving state there is no possibility of scattering backwards. Hence the current is robust. Remarkably, this means that

Current flow is dissipationless, and hence $\rho_{xx} = 0$!

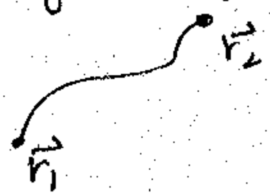
$R_K = \frac{h}{e^2}$ therefore is a Resistance standard, and is called the Klitzing constant.

Now we ask the question: Can we have the quantum Hall effect on the ~~latt~~ lattice? Then we need to ask the question, How do we apply a magnetic field to a tight binding model?

I would like to use the following idea:

I would like to use the continuum
Suppose I have a system in the continuum experiencing a vector potential $\vec{A}$, then an electron in going from $\vec{r}_1$ to $\vec{r}_2$, picks up an additional phase given by

$$\frac{e}{\hbar}\int_{\vec{r}_1}^{\vec{r}_2} d\vec{l}\cdot\vec{A}.$$

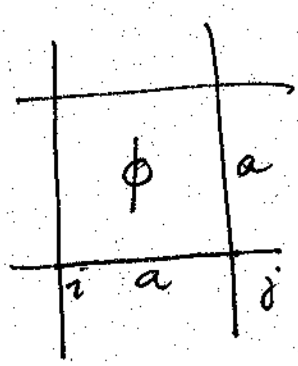


Quite remarkably, the phase picked up depends only on the path and not on the velocity along the path. [Idea (Aside)

$L \simeq \vec{A}\cdot\vec{v}$ $\quad S = \int dt\, L$

$$S = \int \frac{dt\, \vec{v}}{dl} A \Big]$$

So now take a tight binding model on a square lattice and want to pierce a flux of $\phi = Ba^2$ in each plaquette. How do I do this



Well, ~~tod~~ Consider $A_{ij} = -\frac{e}{\hbar}\int_{\vec{r}_1}^{\vec{r}_2} d\vec{l}\cdot\vec{A}$

$t \to t e^{iA_{ij}}$. Now the magnetic field cond $\prod_{\square} e^{iA_{ij}} = e^{i\sum'_{\square} A_{ij}} = e^{i\frac{-e}{\hbar}\oint \vec{A}\cdot d\vec{l}}$

$e^{+i\left(-2\pi\frac{\phi}{\phi_0}\right)}$!

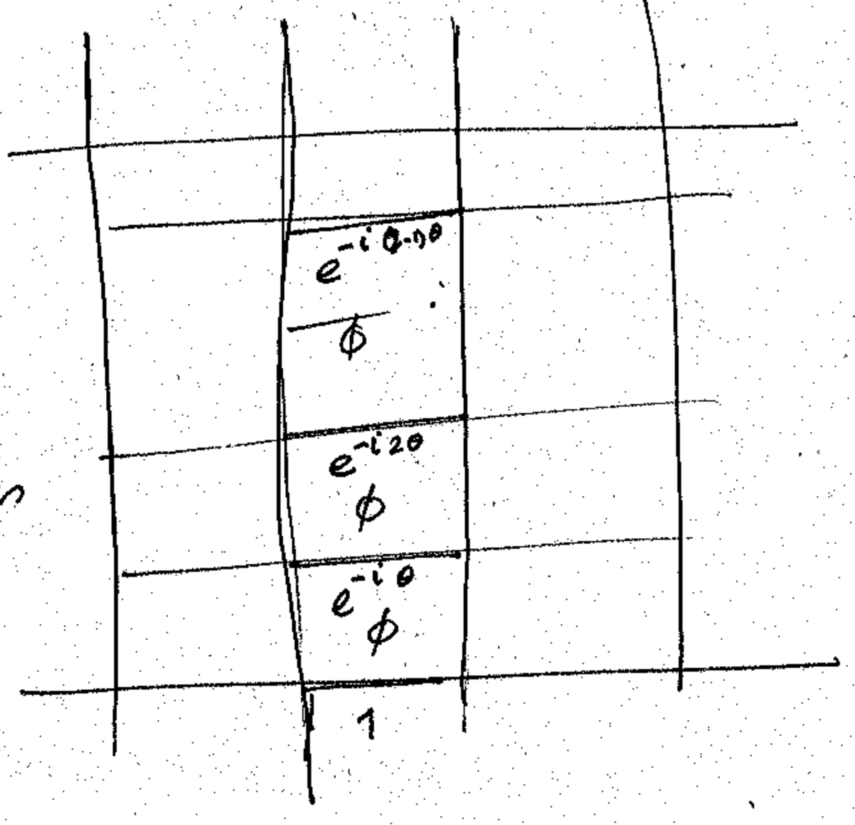
So all we need to do is to pick A_{ij} in such a way that $\sum'_{\square} A_{ij} = 2\pi\frac{\phi}{\phi_0}$

Now suddenly you realize that this is not gauge and moreover we may break the translational symmetry of our TB hamiltonian just as we did in the continuum. Now ~~god~~ Nature is kind to us: if $\frac{\phi}{\phi_0} = \frac{p}{q}$ a rational number then call $\boxed{\theta = \frac{p}{q}2\pi}$

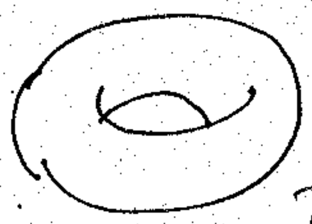
we ~~come~~ save

When $\frac{p}{q}$ = rational this is called the Hofstadter model.

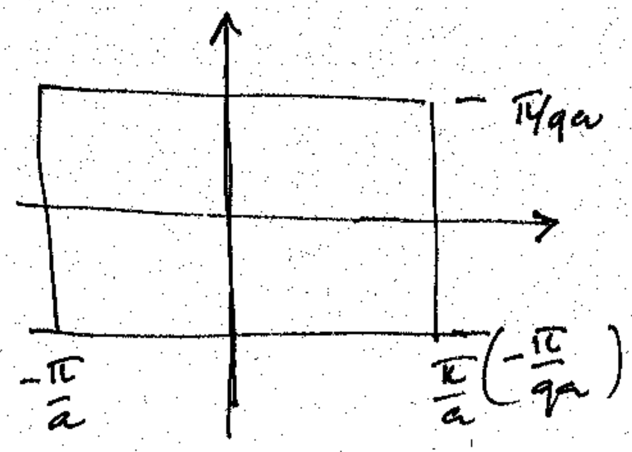
The unit cell ~~box~~ is now ~~spans~~ q sites.



The BZ becomes



Torus!



Let us focus when $\phi = 1$ and q is some number. We will now get q bands for each k in the BZ. Quite remarkably we find that we get bands

Figure

Has bloch states
~~$\psi_{n\vec{k}}$~~ $|u_{n\vec{k}}\rangle$.

"Landau" "like" levels separated by a gap. If the filling of electrons is n/q, then I get an insulator.

Now Thouless-Kohmoto-Nightingale-de Nijs (TKNN) decided to calculate the σ_{xy} for such a system. They showed the following

$$\sigma_{xy} = \sum_{n} \sigma_{xy}^{(n)}$$

Statement is that

$$\sigma_{xy}^{(n)} = \frac{e^2}{h} \underbrace{- \frac{i}{2\pi} \int dk_x dk_y \left[\left\langle \frac{\partial u_{nk}}{\partial k_x} \middle| \frac{\partial u_{nk}}{\partial k_y} \right\rangle - \left\langle \frac{\partial u_{nk}}{\partial k_y} \middle| \frac{\partial u_{nk}}{\partial k_x} \right\rangle \right]}_{C_n}.$$

?

$$\sigma_{xy}^{(n)} = C_n$$

$$C_n = \frac{\cancel{i}1}{2\pi} \int dk_x\, dk_y \underbrace{\Omega^n(k_x, k_y)}_{\text{Berry curvature}}$$

$$-i \left[\left\langle \frac{\partial u_{nk}}{\partial k_x} \middle| \frac{\partial u_{nk}}{\partial k_y} \right\rangle - \left\langle \frac{\partial u_{nk}}{\partial k_y} \middle| \frac{\partial u_{nk}}{\partial k_x} \right\rangle \right] = \Omega^n(k_x, k_y).$$

Now one can show that

$$\Omega^n(k_x, k_y) = \vec{e}_z \cdot \vec{\nabla} \times \vec{\mathcal{A}}^{(n)}$$

$$\frac{\partial}{\partial k_\alpha} |u_{n\vec{k}}\rangle \equiv \left|\frac{\partial u_{n}}{\partial k}\right.$$

$$\mathcal{A}_\alpha^{(n)} = -i \langle u_{nk} | \frac{\partial}{\partial k_\alpha} | u_{nk} \rangle$$

Note that since $\langle u_{nk} | u_{nk} \rangle = 1$, $\mathcal{A}_\alpha^{(n)}$ are <u>real</u>

Now this looks like ~~a~~ a "vector potential" called a (Berry) connection. Ω^n is like a magnetic field!

But you can object to this saying that this makes no sense. Why? You will say,

~~$|u_{nk}\rangle$~~ $|u_{n\vec{k}}\rangle$ are not uniquely defined.

In particular $|\tilde{u}_{nk}\rangle = e^{i\phi(\vec{k})} |u_{nk}\rangle$ is also a possible choice of the Bloch state. No one $\phi(\vec{k})$ is mat

I ~~will say~~

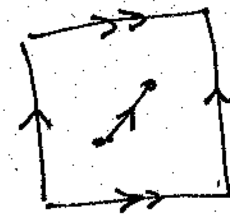
prevents me from using this!

$$\tilde{A}^{(n)}_\alpha = i\langle u_{n\vec{k}} | \frac{\partial}{\partial k_\alpha} | u_{n\vec{k}}\rangle = A^{(n)}_\alpha + \nabla\phi(\vec{k})$$

Then $\cancel{\int \tilde{A}^{(n)}_\alpha}$ ~~Gam~~ Now choose a path on the

torus $1 \to 2$

$$\int_1^2 d\ell_\alpha \cdot \tilde{A}^{(n)}_\alpha = \int_1^2 d\ell_\alpha \cdot A^{(n)}_\alpha + (\phi(2)-\phi(1))$$

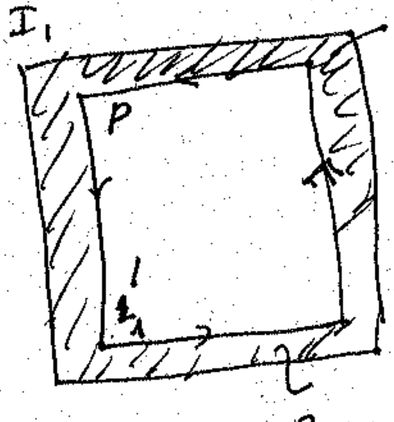
= 

Based on this we can interpret

$\int_1^2 d\ell_\alpha A^{(n)}_\alpha$ to be the phase picked up by the Bloch function $|u_{n\vec{k}}\rangle$ in going from $1 \to 2$

This is what we need.

Ok! Let us look at $C_n = \frac{1}{2\pi}\int_{\text{BZ}} d^2k\, \Omega^n_{z}(\vec{k})$



$$\frac{1}{2\pi} C_n = \boxed{\frac{1}{2\pi}\int_B d^2k\, \Omega^n_{z}(\vec{k})}$$

$$\phi(1')-\phi(1) = \frac{1}{2\pi}\int_P d\vec{k}\cdot A^{(n)}(\vec{k})$$

$2(\pi\ \ )$

$$= -\frac{1}{2\pi}\int_{B'} d^2k\, \Omega^n_{z}(\vec{k}) = 2\pi I_2$$

$B' \cup B = BZ$

Now, ~~$|u_n\rangle \to e^{\frac{i}{2\pi}}$~~

$|u_{n1}\rangle \to e^{i\,2\pi I_1}|u_{n1}\rangle$ when looked at from B.

But $|u_{n1}\rangle \to e^{+i\,2\pi I_2}|u_{n1}\rangle$. when taken along the <u>same path</u>. If I now let $B' \to \phi$

then $I_2 \to 0$ $|u_{n1}\rangle \to |u_{n1}'\rangle$

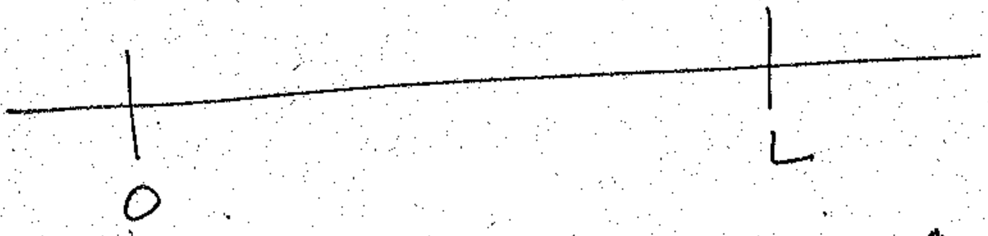
~~ϕ~~ $\Delta\phi' = 0$ but ~~$\Delta\theta$~~

$\Delta\phi = 2\pi C_n$! The only way to reconcile these two things is that C_n <u>has</u> to be an integer! because $e^{i0} = e^{i2\pi n}$!

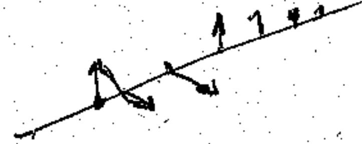
~~Let me show you a~~ This is actually a ~~deep~~ beautiful topological result. Let me show you another. Take a string and define $\psi(x) = e^{i\theta(x)}$ a unimodular complex number.

$e^{i\theta(L)} = e^{i\theta(0)}$

$\psi(x) = \psi(L)$



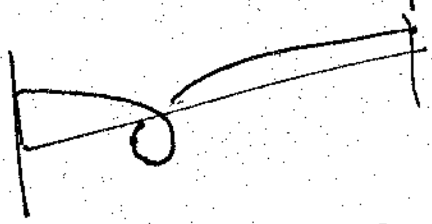
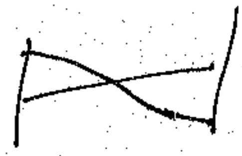
Now define $\tilde{A}(x) = -i\,\psi^*(x)\frac{\partial}{\partial x}\psi(x)$



$= \nabla\theta(x)$.

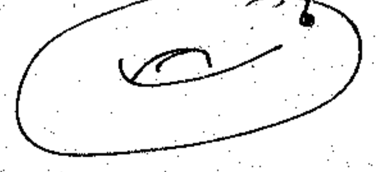
$\int_0^L dx\,A(x) = \theta(L) - \theta(0) = 2\pi n$ when ...

Now we see that the functions ψ fall into different topological classes! ~~Ψ~~ with the same n can be smoothly "deformed" to the other

~~bt~~  $\neq$ 

Two ψ's with the same n are topologically equivalent. — called a topological class!

Now the Chern number is quite similar at every point on the torus, I am attaching a Bloch state. The Bloch state can wind and twist around in the BZ ~~until that gives some~~ but it has to come back to itself. that is why it is an integer.



$$\sigma_{xy} = \frac{e^2}{h} \sum_{n \text{ occupied}} \sigma_{xy}^n = \frac{e^2}{h} \sum_{n \text{ occupied}} C_n$$

Suddenly we realize that the response function of a system is determined only by its topological class! And hence an integer. Now

Suppose I add some disorder..

~~I do~~ Note that ~~the~~ for the Chern number of a band to be defined properly, the band should not touch other.

For the Hofstadter model

$$\sigma_{xy} = \frac{e^2}{h} \sum_{n\,\text{occupied}}' c^n$$

Question what $\sum'_{\text{all band}} c^n$ $\longrightarrow$ $\boxed{\text{Fig Hof Chern Hullify}}$

Built we know from Laughlin argum that σ_{xy} is realled to number of edge states!

~~Now Hat~~ Are there edge states in the Hofstadter problem! Yes $\boxed{\text{Fig.}}$

Hatsugai's result $C_n = \cancel{N^{out}_{edge} - N^{in}_{edge}}$!

$E^{out}_n - E^{in}_n \hookrightarrow E^{in}_n$ (on an edge)

$$\Rightarrow E^{in}_{n\,\text{lowest}} = 0, \quad E^{out}_{last} = 0$$

$$\Rightarrow \sigma_{xy} = \frac{e^2}{h} \, E^{out}_{n\,\text{highest}} \ !$$

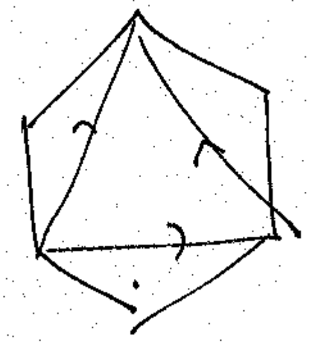
We now see the key connection between chern number and edge states.

Ok: We now ask: Can we get quantum Hall effect without a magnetic field?

Back to graphene and enter Haldane:

- idea add a ~~second~~ next neighbour hopping that gives

Idea is to introduce alternate flux

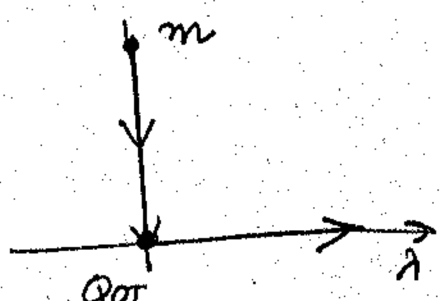


$$\mathcal{H}_{Haldane} = \mathcal{H}_{graphe} + \sum_I i\lambda \left(c^\dagger_{I+2A}\, c_I + h.c.\right)$$

This is the Haldane model and has robust edge states.

Bottom band has chern number 1 since there is 1 edge state - we have quantum Hall effect!

When we go to right we encounter a QCP at the ~~New~~ Graphene point. The gap has to close to achieve this.

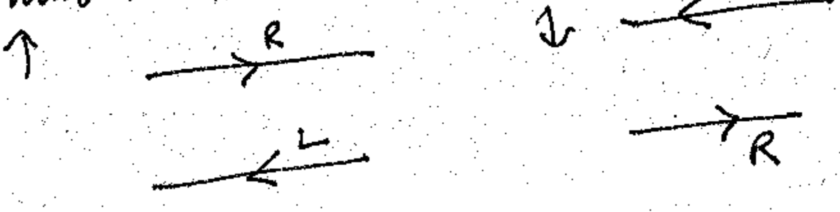


Now you ~~can say~~ ask How to restore TR symmetry?

$$\mathcal{H}_{Hal} = \mathcal{H}_{grp}^{(spinful)} + \sum_I i\sigma\lambda \left(c\, c^\dagger_{I+2B\,\sigma}\, c_{I\sigma} + h.c.\right)$$

The Hamiltonian now has Time reversal symmetry!

Note that S_z is conserved. Spin is a good quantum number. and the spin↓ band is a TR copy of the



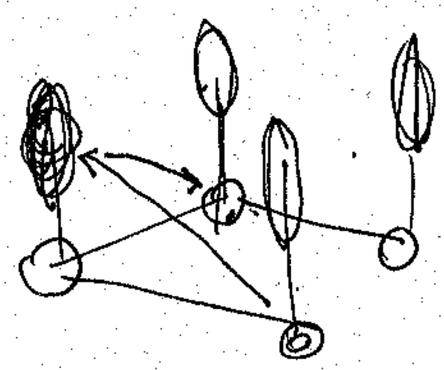
Note that the edge states are quite robust.

Now this is a somewhat contrived model: Can we get such things happening in real materials? Answer is Yes! BHZ (Bernevig, Hughes, Zhang) model using spin orbit coupled system with an inverted Band structure.

What is spin orbit coupling? In addition to what lies there is an $\vec{L}\cdot\vec{S}$ term in the potential.

so if we have usual $|p_x \uparrow\rangle$, $|p_y \uparrow\rangle$ etc,

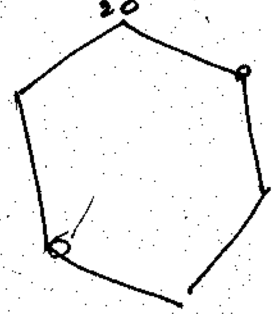
$L\cdot S$ terms will make the local orbitals $|p_x + i p_y\rangle$



Without (SO) I will not be able to do this.

It is still easier to discuss spin orbit interaction in the honeycomb lattice. Kane-Mele model. In graphene they introduced an additional term

$$\mathcal{H} = \mathcal{H}_g + i\lambda_{SO} \sum_{i\delta} c_i^\dagger \vec{S}\cdot\vec{d}\;\vec{S}$$



So in general spin can change when it hops!

We get four bands and a band structure quite similar to massive graphene

In general the Hamiltonian now will look like

$$\mathcal{H} = \sum_{\substack{k \\ a,b \\ \sigma\sigma'}} \underbrace{\mathcal{H}^{ab}_{\sigma\sigma'}(\vec{k})}_{4\times4 \text{ matrix}} c^{\dagger}_{ka\sigma} c_{kb\sigma'}$$

We get 4 bands and the lower two bands are of interest.

$$\mathcal{H}\,|u_1(\vec{k})\rangle = E_1(\vec{k})\,|u_1(\vec{k})\rangle$$

$$\mathcal{H}\,|u_2(\vec{k})\rangle = E_2(\vec{k})\,|u_2(\vec{k})\rangle$$

$$\Theta\,|u_1(\vec{k})\rangle = e^{-i\chi(\vec{k})}\,|u_2(-\vec{k})\rangle$$

$$\Theta\,|u_2(\vec{k})\rangle = -e^{i\chi(-\vec{k})}\,|u_1(-\vec{k})\rangle.$$

From this one can immediately show that

$$\Omega_1(\vec{k}) = -\Omega_2(\vec{k}) \Rightarrow C_1 + C_2 = 0$$

or $C_1 = -C_2$ since TR symmetry,

$\Rightarrow \sigma_{xy} = 0$! (Not surprising since TR symmetry)

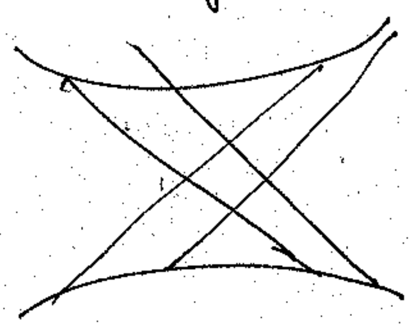
Now look at the edge state structure, you see that there are two distinct types one where TRIMS are connected pairwise and other where they are connected alternately to the higher band. Clearly the first type is trivial while the second type is nontrivial as is like the KM model. Note that the edge states with ~~odd~~ are number are protected

Note now that. So what we get here is a time reversal polarization current $\sim \frac{e^2}{h} * \text{integer}$ "The quantum "time polarization current effect'!

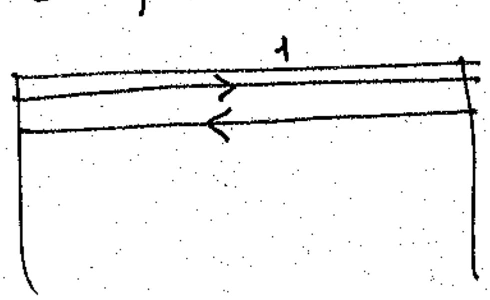
$\sigma_{xy}^{TR} \neq 0$ in a topological insulator

One <u>can</u> distinguish the topological insulator and an ordinary insulator via $K^{\alpha\beta}$ like this!

Now you can legitimately ask why can I not have



Something like this? One can show that



an odd ~~number~~ number of bands is stable against any TR invariant perturbation, while even number of bands are not.

Thus number of bands ($\sum_j C_j$) of the chen number of the occupied bands. For a 4 band system it is C_1. If C_1 is odd we get TI else

$\boxed{\nu = \begin{cases} +1 & C_1 = \text{even} \\ -1 & C_1 \text{ odd} \end{cases}}$ is the Z_2 index.

~~Pl Pro~~
What happens in 3D?
It turns out that this way to distinguish a real 3D topological insulator and an ordinary one is to see wealth the system as Magneto electric polarizability.

$$= \frac{\theta}{4\pi} \vec{E}\cdot\vec{B}$$

In the electromichue feed duty.

$$\theta = \left.\begin{matrix} 0 \\ \pi \end{matrix}\right\} \text{ for TR invariance (defined only up to } 2\pi)$$

$$\theta = -\frac{1}{4\pi}\int_{BZ} d^3\vec{k}\, \mathrm{Tr}\left[\cancel{\vec{A}\times(\vec{A}\times\nabla\vec{A})}\ \vec{A}\times(\vec{\nabla}_k\vec{A}) - i\frac{2}{3}\vec{A}\cdot(\vec{A}\times\vec{A})\right)$$

where $A_{mn} = -i\langle u_{mk}|\frac{\partial}{\partial k}|u_{nk}\rangle$ is the connection matrix ~~and~~ whose size = the number of occupied bands. For topological reasons this turns out to be ~~…~~ 0 or π. 0 is ordinary