Untwisting Twisted Matter
An Invitation to Topological Phases of Electrons

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Overview

- Background – Electronic phases: A brief excursion of quantum condensed matter physics
- Electronic Phases – A quick illustration of “topology”
- Electronic Phases - A summary of what we know today
- Open questions
- ...
- Very brief intro to talks by Mandar and Zlatko

Topological Matter Matters

Photo: A. Mahmoud

David J. Thouless

Prize share: 1/2

Photo: A. Mahmoud

F. Duncan M. Haldane

Prize share: 1/4

Photo: A. Mahmoud

J. Michael Kosterlitz

Prize share: 1/4

The Nobel Prize in Physics 2016 was awarded with one half to David J. Thouless, and the other half to F. Duncan M. Haldane and J. Michael Kosterlitz *"for theoretical discoveries of topological phase transitions and topological phases of matter"*.

Phases of Electrons

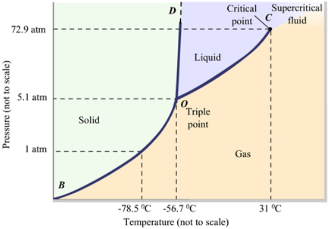
Matter...and its Phases

- Matter broadly appears in three distinct phases (at human scales)



(Internet)

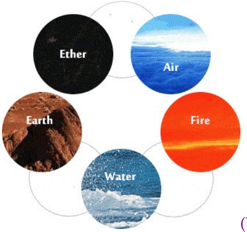
- ...gas, liquid and solid



(Phase Diagram of CO_2, source:Internet)

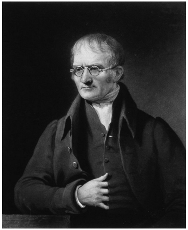
Whence Phases?

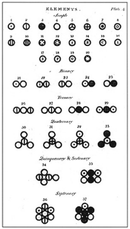
- Ancient wisdom...the panchabhootas...



(Wikipedia)

- ...to ideas of Dalton...the atomic hypothesis (early 1800s)

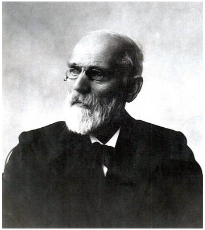
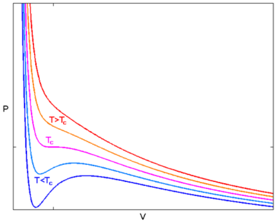




(Wikipedia)

Atomic Theory...to Theory of Phases

- van der Waals (later part of 1800s) showed how liquids and gases can arise from the same constituent atoms/molecules...

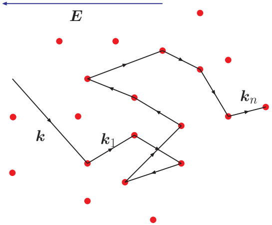
...offering a framework to understand gas, liquid and solid phases

- Interactions between constituent atoms (in large numbers) can lead to different phases
- Puzzle: Why are there insulators and conductors?

Electrons...and Electronic Phases

- Key milestone: Discovery of the electron (Thomson, 1890s)...

- ...to the first theory of electronic phases – Drudé (1900)

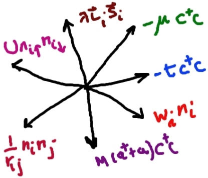


with remarkable success...resolution of a 50 year old puzzle – the Wiedemann-Franz law ($\frac{\kappa}{\sigma T}$ = universal number)...

- **But**...with a confounding new puzzle: Drude predicts $C_V = 3R + \frac{3}{2}R = \frac{9}{2}R$ at "high" temperatures...measured $C_V = 3R$!

Quantum Condensed Matter Physics – Bird's Eye View

- Resolution of the puzzle: Quantum mechanics
- Electrons in materials "see" many things...

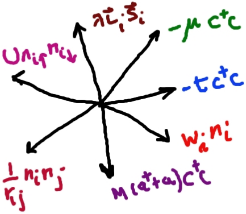


...space of all possible Hamiltonains (includes kinetic energy, spin-orbit coupling, Coulomb interaction, interaction with the lattice, disorder etc..)

- Quantum condensed matter physics aims to study and classify the phases of many electrons (many identical particles, in general)
- Many of the modern technologies from cell phones to night vision goggles owe much to this area of physics!

Taking Stock...

- Quantum theory of many electrons – offers insights into insulators and metals...



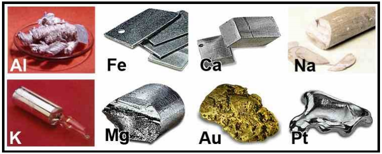
Traditional ideas of classifying many-fermion phases

- Symmetry: A magnet breaks spin-rotation symmetry (Landau)
- "Properties": Metals and insulators

Distinct phases are separated by *phase transitions*

Summary of a Grad Course!

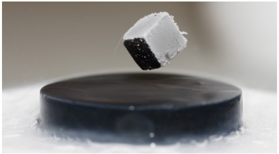
- Gist of our understanding...*essentially* captured by **four** states
- Gapless (single fermion excitation)[1]



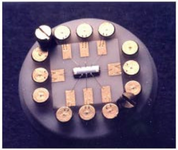
1. The filled Fermi sea

- Gapped states

2. Band insulator



3. BCS superconductor



4. Filled Landau level

[1] All images are from the internet

Current Status

- Recent developments – **two** ideas (in addition to symmetry, properties etc.)
  1. **Topology:** Hinted by the quantum hall effect
  2. **Entanglement:** How "complicated" is the state?

  Focus of this talk: "Topology"
- Recent developments – Complete "topological" classification of gapped (non-interacting) many fermion systems

Cartan\\d	0	1	2	3	4	5	6	7	8	
Complex case:										
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
Real case:										
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\cdots$
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$\cdots$
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

(Kitaev (2009), Ryu et. al. (2010), Ludwig:1512:08882)

- The tenfold way!

Electronic Phases

...What and Why Topology?

Essential Quantum Ideas

- Quantum mechanics
  - Kinematics : Identical particles are indistinguishable
    Fermions obey Pauli's principle
    For two fermionic particles

$$\Psi(x_2, x_1) = -\Psi(x_1, x_2)$$

  - Dynamics "encoding" the Heisenberg uncertainty principle
- Immediate goal: Construct simplest electronic phases with focus on *non-interacting* systems

Warm Up: Atomic Structure – Few Electrons

- Carbon atom: Six electrons in the nuclear potential with charge +6
- Electronic structure (wave function) (NCERT – 11th Class Chemistry)

$$\Psi_C(x_1,\ldots,x_6) = \begin{vmatrix} 1s_\uparrow(x_1) & 1s_\uparrow(x_2) & 1s_\uparrow(x_3) & 1s_\uparrow(x_4) & 1s_\uparrow(x_5) & 1s_\uparrow(x_6) \\ 1s_\downarrow(x_1) & 1s_\downarrow(x_2) & 1s_\uparrow(x_3) & 1s_\downarrow(x_4) & 1s_\downarrow(x_5) & 1s_\downarrow(x_6) \\ 2s_\uparrow(x_1) & 2s_\uparrow(x_2) & 2s_\uparrow(x_3) & 2s_\uparrow(x_4) & 2s_\uparrow(x_5) & 2s_\uparrow(x_6) \\ 2s_\downarrow(x_1) & 2s_\downarrow(x_2) & 2s_\uparrow(x_3) & 2s_\downarrow(x_4) & 2s_\downarrow(x_5) & 2s_\downarrow(x_6) \\ 2p_{X\uparrow}(x_1) & 2p_{X\uparrow}(x_2) & 2p_{X\uparrow}(x_3) & 2p_{X\uparrow}(x_4) & 2p_{X\uparrow}(x_5) & 2p_{X\uparrow}(x_6) \\ 2p_{Y\uparrow}(x_1) & 2p_{Y\uparrow}(x_2) & 2p_{Y\uparrow}(x_3) & 2p_{Y\uparrow}(x_4) & 2p_{Y\uparrow}(x_5) & 2p_{Y\uparrow}(x_6) \end{vmatrix}, \ (x \equiv (\boldsymbol{r}, \sigma))$$

...Slater determinant wave function

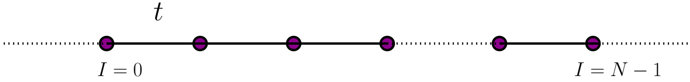
- Structure known for any atom...the periodic table

Group→ ↓Period	1	2	3		4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1 H																		2 He
2	3 Li	4 Be												5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg												13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca	21 Sc		22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr	39 Y		40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	57 La	*	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	89 Ac	**	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Nh	114 Fl	115 Mc	116 Lv	117 Ts	118 Og
				*	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu	
				**	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr	

- Question: Analogous table of many (10^{23}) electron phases?

Electronic Liquid Phase – Metal

- Simple model of a metal (spinless electrons)...tight binding model



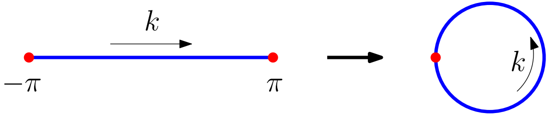
...N-site chain with periodic boundary conditions

$$H = -t \sum_I |I+1\rangle \langle I| + \text{h.c.}$$

- Diagonalized by crystal momentum k

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_I e^{-ikI} |I\rangle\,, \qquad H = \sum_k \varepsilon(k) |k\rangle \langle k|\,, \quad \varepsilon(k) = -2t\cos k$$

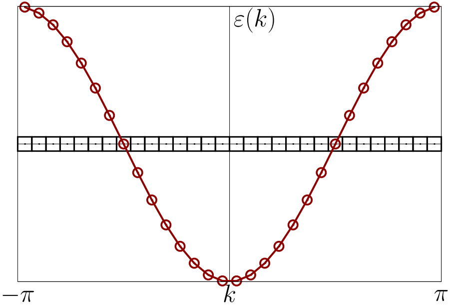
- Crystal momentum k lives in the Brillouin Zone (BZ)...$k \in [-\pi, \pi]$ with $-\pi$ and π identified...one dimensional torus



- In d-dimensions k in T^d, the d-torus

Electronic Liquid Phase – Metal

- Filling of electrons 1/2 per site, need to fill $N/2$ electrons
- In an N-site chain, there are N distinct k points
  $k = -\pi, -\pi + \frac{2\pi}{N}, -\pi + 2\frac{2\pi}{N}, \ldots, \pi - \frac{2\pi}{N}$ with $\Delta k = \frac{2\pi}{N}$



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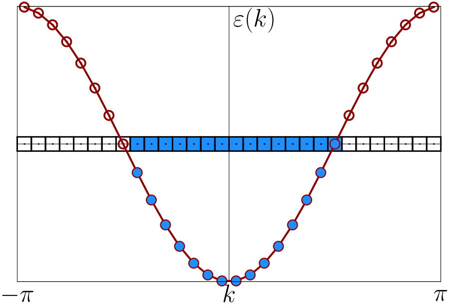

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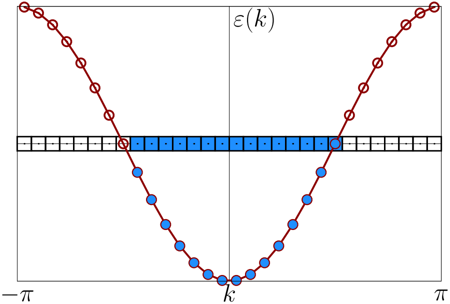
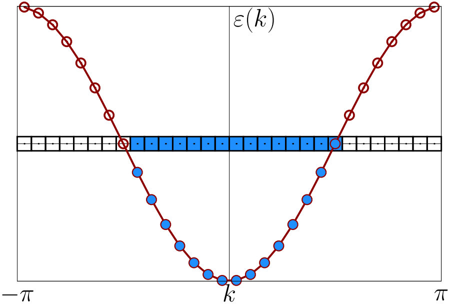


- Ground state is the filled Fermi sea (filled "pigeon holes" make up the Slater determinant wave function)

Electronic Liquid Phase – Metal

- Gapless excitations in the thermodynamic limit ($N \to \infty$)



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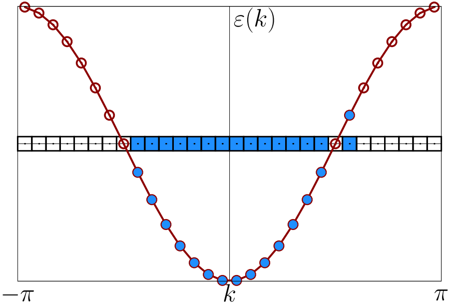

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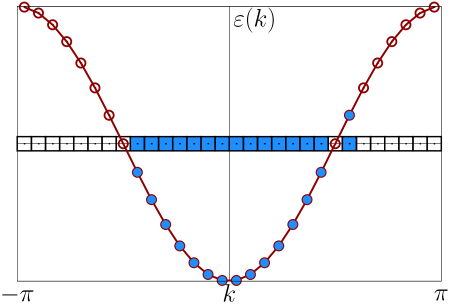

…excitation energy vanishes

$$\Delta E \sim \Delta k \sim \frac{2\pi}{N} \to 0$$

- Metal: small stimulus can produce finite responses – liquid state
- Fermi energy ($\sim t$), is much larger than temperature T, resolving the specific puzzle of the Drude theory

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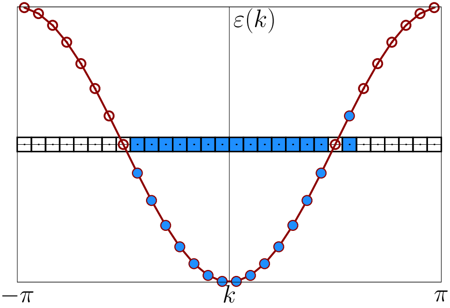

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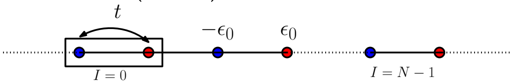
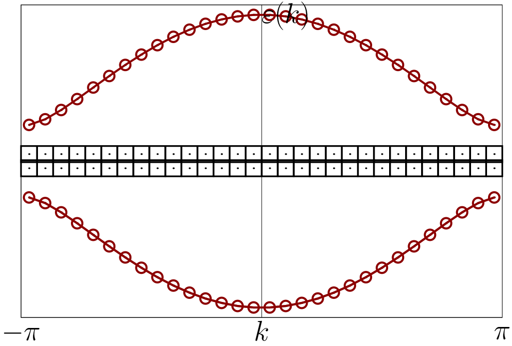
- Puzzle...Insulators?

..and Insulators

- Poor man's insulator (NaCl)



- *Two* bands (with separate pigeon holes) separated by a band gap $\sim \epsilon_0$ ...there are no electronic states in the energy gap



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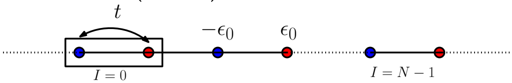

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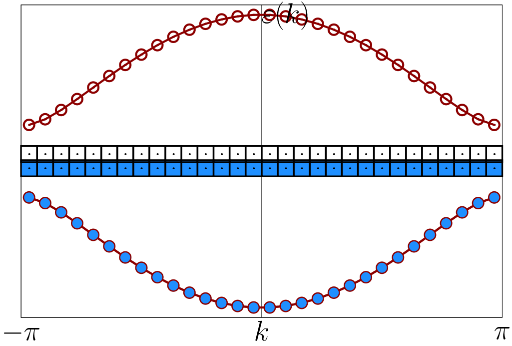

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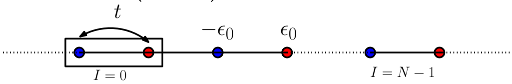

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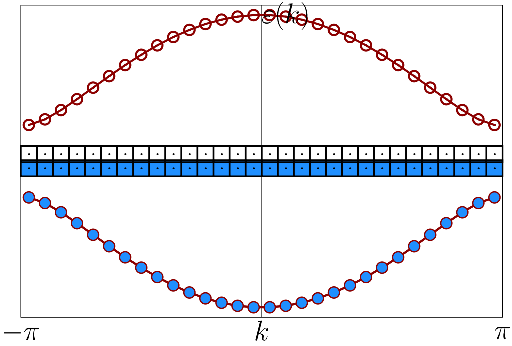
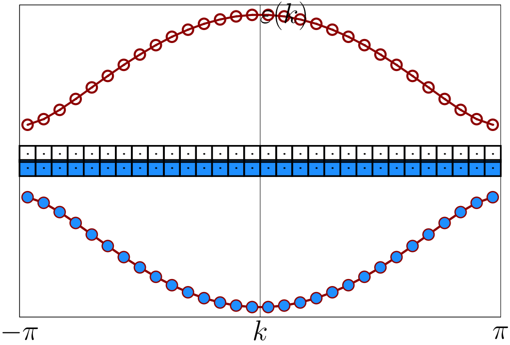


- At $1/2$ electron per site, ground state is the filled "valance band" ("conduction band" is empty)

Insulators: "Electronic Solid Phase"

- Excitations are gapped



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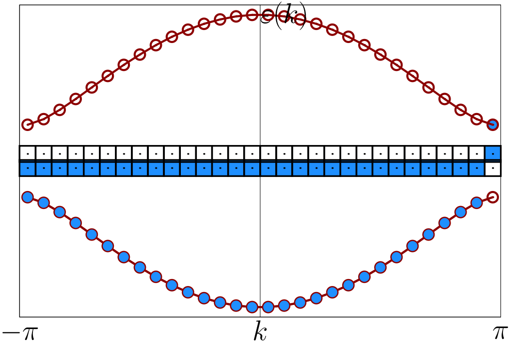

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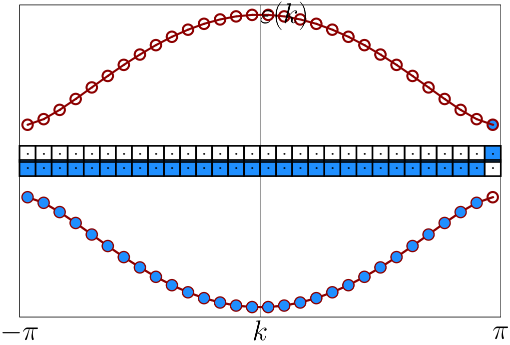

- Insulators play a pivotal role in electronics – *doped* insulators (semiconductors) are behind much of modern technology

Insulators: “Electronic Solid Phase”

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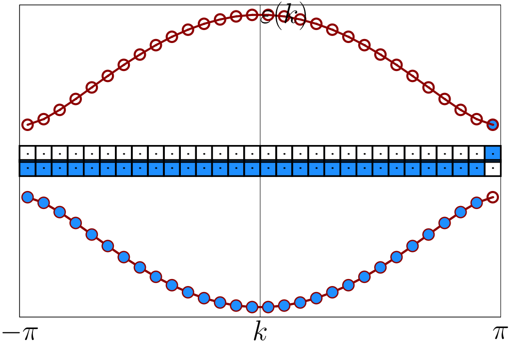
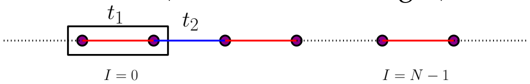


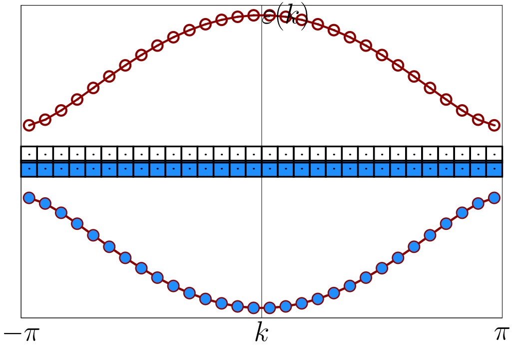
- Insulators play a pivotal role in electronics – *doped* insulators (semiconductors) are behind much of modern technology
- Question: Is this the only possible insulating phase?

Topology of Electron Phases – Whetting the Appetite

- A different insulator – SSH (Su-Schrieffer-Heeger) model

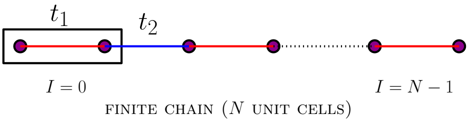


- Band structure looks qualitatively identical to the NaCl system (gap $\sim |t_2 - t_1|$)

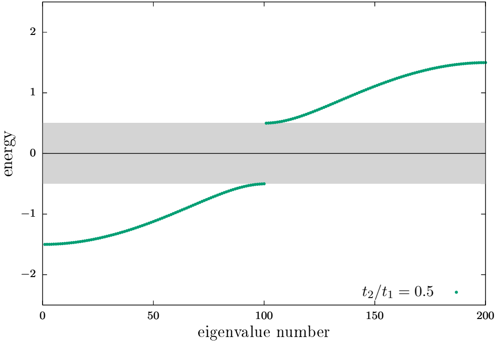


- *Cannot* distinguish between NaCl and SSH by looking at the bulk band structure

Topology...Revealed in a *Finite* Chain



- Energy eigenvalues
  - No energy levels in the gap for $t_2 < t_1$ (NaCl is similar)



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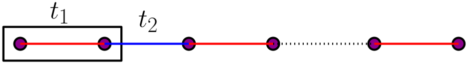

FINITE CHAIN (N UNIT CELLS)

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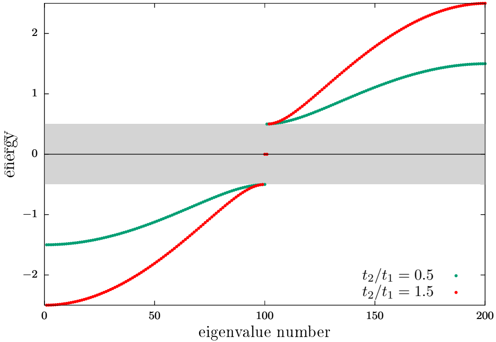
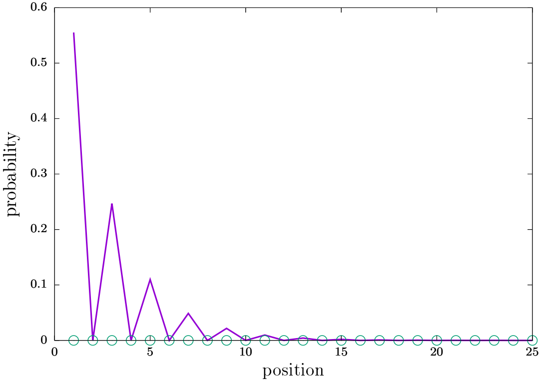


  - Two *zero energy states* when $t_2 > t_1$

Topology...on the *Edge*

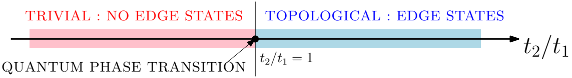
- For $t_2 > t_1$ the zero energy states are *edge states*...they are eigenstates localized near the edges ("surfaces") of the finite sample



- Insulating phase for $t_2 > t_1$ has a distinguishing character – presence of edge states

Phases Distinguished by Topology...and Phase Transitions

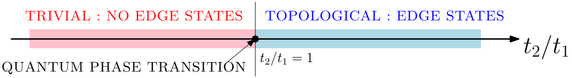
- Phase diagram



- Can drive a quantum phase transition by tuning t_2

Phases Distinguished by Topology...and Phase Transitions

- Phase diagram



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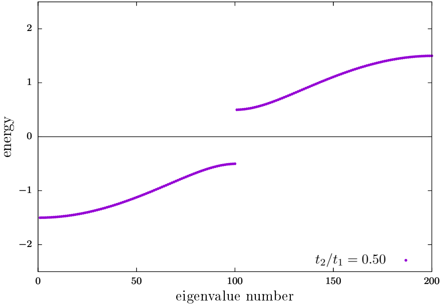

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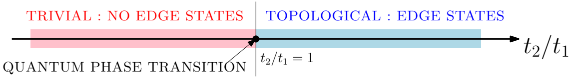

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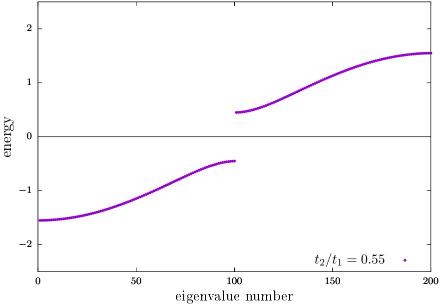

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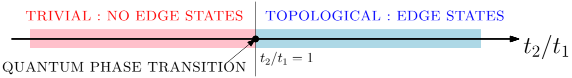

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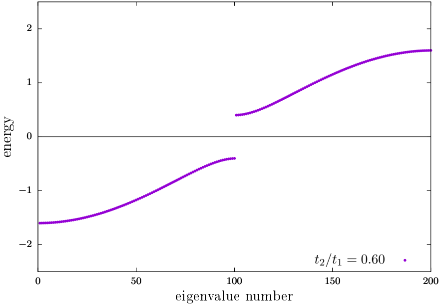

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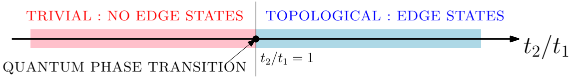

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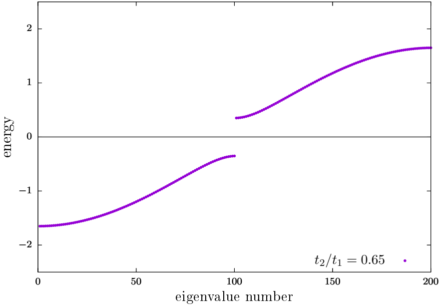

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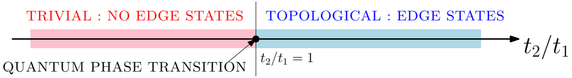

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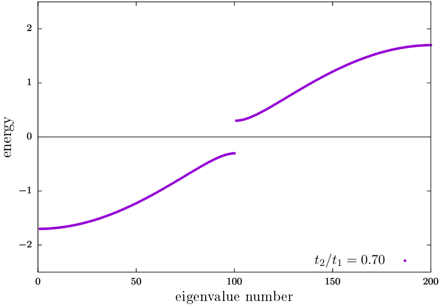

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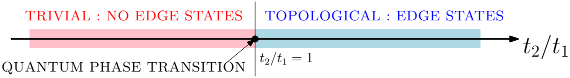

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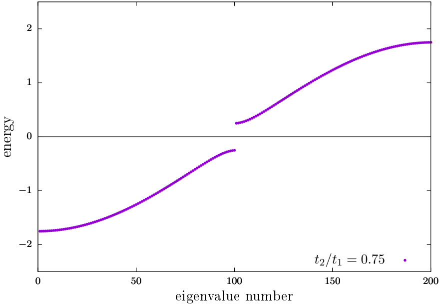

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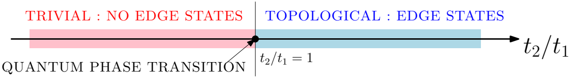

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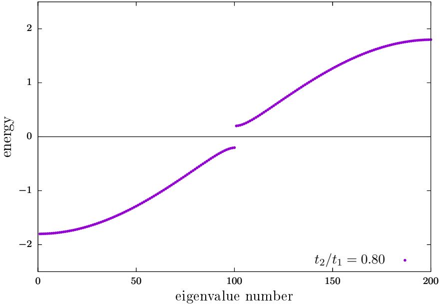

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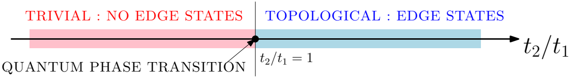

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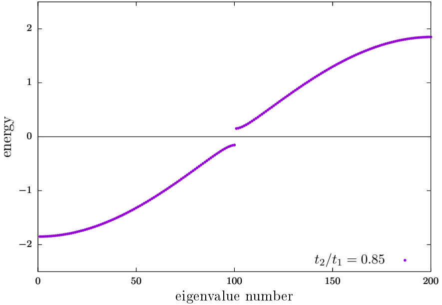

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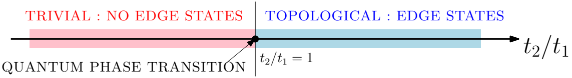

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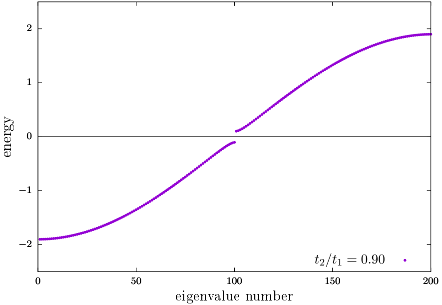

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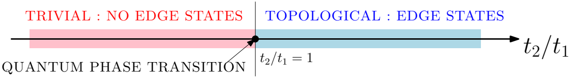

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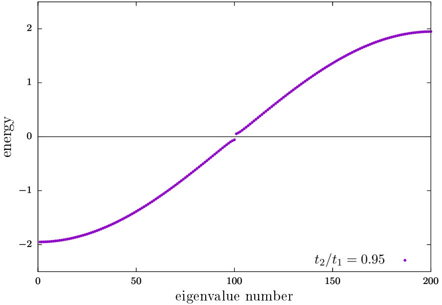

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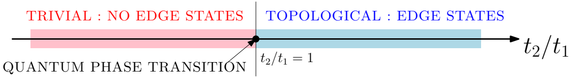

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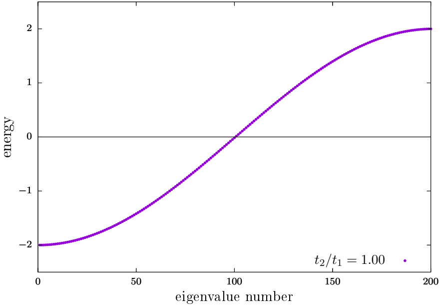

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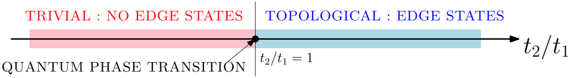

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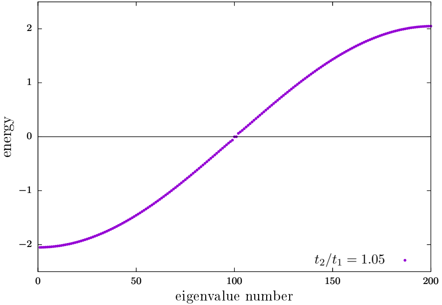

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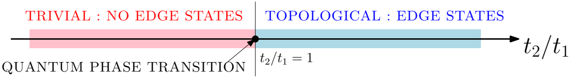

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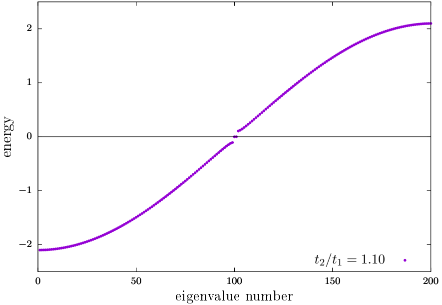

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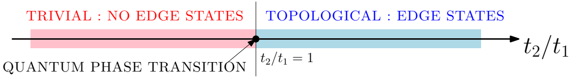

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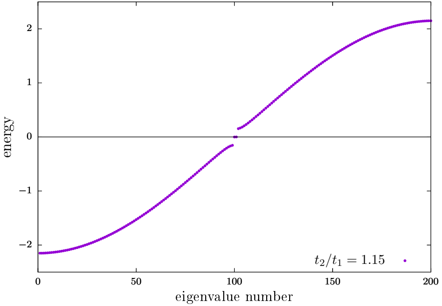

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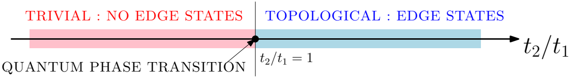

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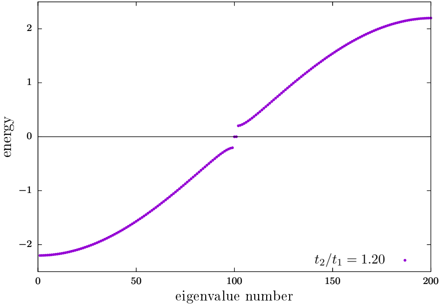

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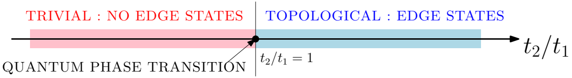

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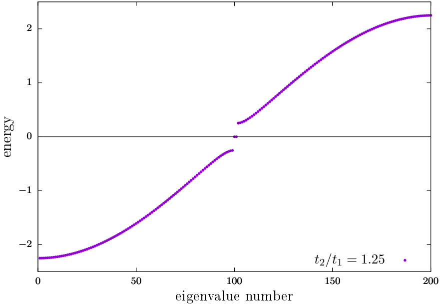

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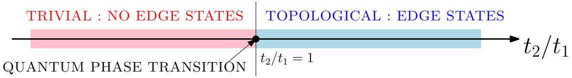

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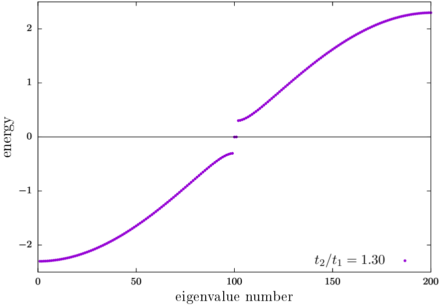

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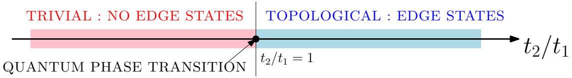

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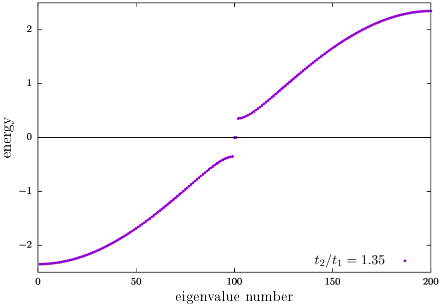

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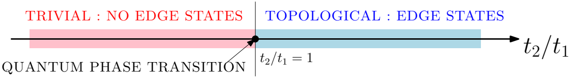

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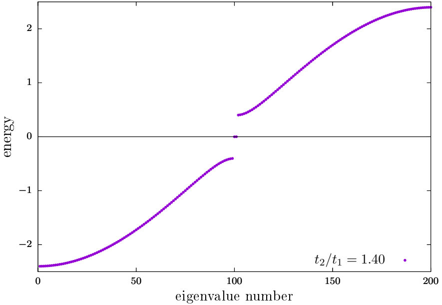

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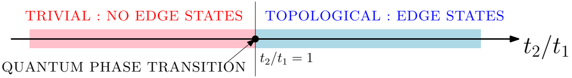

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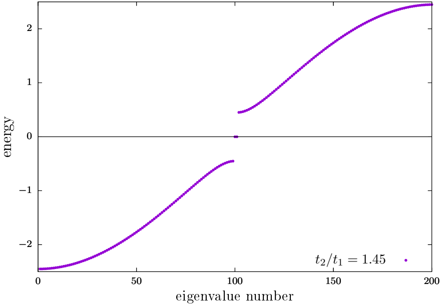

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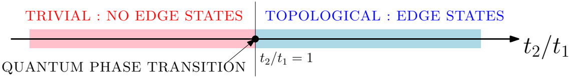

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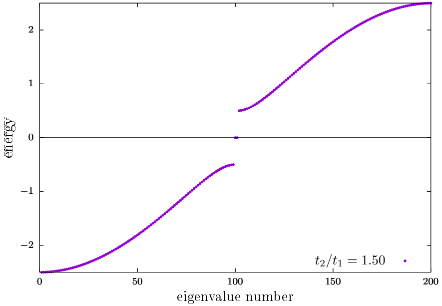

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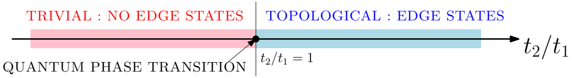

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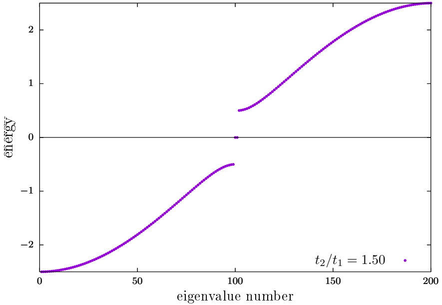

Insulating phase for $t_2 > t_1$ has a distinguishing character – presence of edge states

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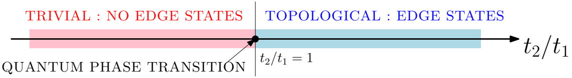

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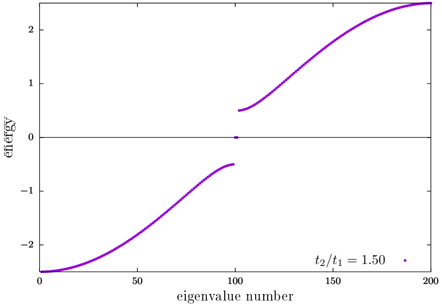
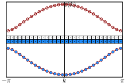


Insulating phase for $t_2 > t_1$ has a distinguishing character – presence of edge states

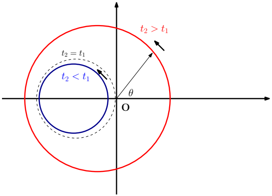
- **Question: Where *IS* the topology?**

Topology of Electron Phases

- VB wave function $\begin{pmatrix} 1 \\ e^{\mathbb{i}\theta(k)} \end{pmatrix}$ with $\theta(\pi) = \theta(-\pi) + 2\pi n$, Key: The state at k can be thought of as a two dimensional unit vector
- Ground state is filled valance band



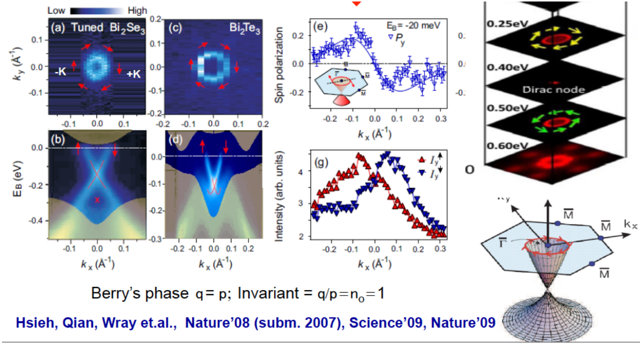
  can be viewed as an endless ribbon(Demonstration)
- For $t_2 > t_1$ the ground state ribbon is "twisted"



- The ground state of $t_1 < t_2$ cannot be deformed to that of $t_2 > t_1$ without closing the gap (tearing)
- **Topology is encoded in the *twist* of the many particle wave function**

Topological Insulators

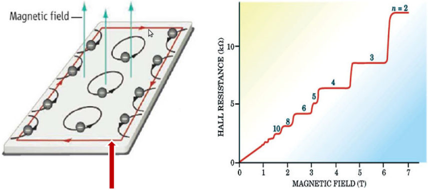
- Such physics can be realized in higher dimensions
- Topological insulators:Insulators in bulk, metals on the surface
- Realized material systems with strong *spin orbit coupling*



(Pinceton group, BiSb system)

Topological Insulators for Technology

- Insulating bulk and conducting surfaces offer *many* possibilities
- Surface state transport is dissipation-less and quantized...more energy efficient devices
- Useful in metrology (standards), resistance quantized to better than one part in a billion (von Kiltzing (2012))



- Topological insulators combined with other systems such as magnets and superconductors can lead to even more interesting physics (..most recent: discovery of Majorana modes using topological phases (Science, July 21, 2017))...useful in qauntum computing
- Challenge: Finding "good" topological insulators, need topology at room temperature

Summary

This talk (Ask me in person for references/review articles)

- Key message: Ideas of topology and entanglement are crucial in understanding/classifying phases of many fermions
- Tenfold way of classifying fermionic systems

Cartan\\d	0	1	2	3	4	5	6	7	8	
Complex case:										
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
Real case:										
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\cdots$
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$\cdots$
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

Instrinsic symmetries of fermions realize “symmetric spaces”, classification has nice mathematics : ask in the discussion session, if interested

Open issues

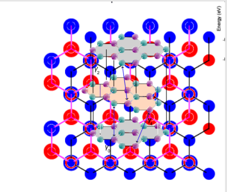
- Topological classification in presence of interactions ...exciting times in condensed matter physics
- Finding and using topological phases – more efficient electronics to quantum computers

This Session

Upcoming Talks

- Zlatko: Interaction effects: How they influence patterns of entanglement to produce fascinating new excitations

- Mandar: Berry phases/topology in graphene systems



Topology/Entanglement: *Making Physics Great Again*

Topology/Entanglement: *Making Physics Great Again*

Current Understanding/Status

- Need two new ideas topology and entanglement to describe phases of many degrees of freedom
- For short ranged entangled phases, gapped fermionic phases are completely classified
- Several topological materials have been discovered
- Interacting systems have been explored, novel excitations probed

Open Questions/New Directions

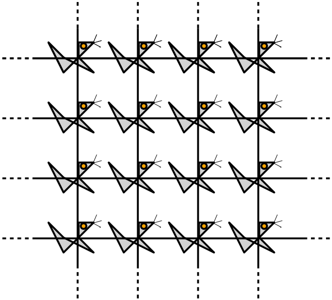
- Theory: Classification of phases based on topology/entanglement (connections to mathematics: classification of vector bundles, group cohomology, etc...)
- Materials Science/Chemistry: Development of new topological materials
- Devices/Mesoscopic Systems: Possible use of topological phases to make, for example, platforms for quantum computation
- Light/Photonics: Topological photonics
- Mechanics: Mechanical systems with topological excitations
- ...
- Your idea here

Details

- Kitaev, AIP Conference Proceedings, 1134, 22 (2009).
- Ryu, Schnyder, Furusaki, and Ludwig, New Journal of Physics, **12**, 065010 (2010).
- Chiu, Teo, Schnyder, Ryu, arXiv:1505.03535
- Ludwig, arXiv:1512.08882

From GFS to Lattice

- Make a lattice out of GFSs in d-dimensions



- $\Psi_I^\dagger$ – fermion operators at site I
- Non-ordinary symmetries implemented "locally" (simplest case)

$$\mathscr{U}\Psi_I^\dagger\mathscr{U}^{-1} = \Psi_I^\dagger\mathbf{U} \quad \text{or} \quad \Psi_I^T\mathbf{U}$$

1

- Hamiltonian

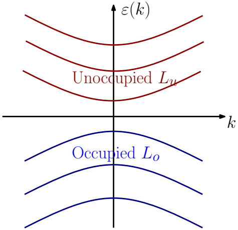
$$\mathscr{H} = \sum_{IJ}\Psi_I^\dagger\mathbf{H}(I,J)\Psi_J$$

Bands etc.

- Bloch picture

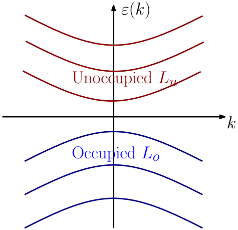
$$\mathscr{H} = \sum_{\boldsymbol{k}} \Psi^\dagger_{\boldsymbol{k}} \mathbf{H}(\boldsymbol{k}) \Psi_{\boldsymbol{k}}$$

  $\boldsymbol{k}$ is in the 1st Brillouin zone – T_d, the d-torus
- Symmetries constrain $\mathbf{H}(\boldsymbol{k})$, e.g., time reversal implies $\mathbf{H}(-\boldsymbol{k}) = \mathbf{H}^*(\boldsymbol{k})$...i.e., symmetries determine the "character" of the Bloch states
- Focus on gapped systems...ground state obtained by "filling" bands below the chemical potential $\mu(=0)$



Ground State...and Topology

- Ground state $|GS\rangle$ – filled bands below μ



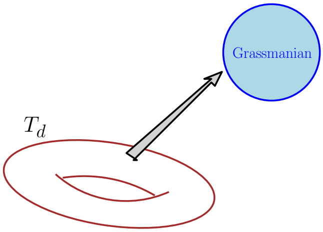
- Two systems $\mathscr{H}_1$ and $\mathscr{H}_2$ in the same symmetry class are topologically equivalent if the there is a continuous deformation of the Hamiltonian from $\mathscr{H}_1$ to $\mathscr{H}_2$ that takes $|GS_1\rangle$ to $|GS_2\rangle$ without closing the gap in the deformation process
- Key question: Given a symmetry class, how many topologically equivalent subclasses are there in d-dimensions?

Topology of Ground States

- Focus on class A: Ground state at any $\boldsymbol{k}$ is a Slater determinant of the occupied Bloch states
- "Gauge freedom" in describing this Slater determinant has to be removed – ground state at $\boldsymbol{k}$ is an object that looks like

$$\frac{U(L)}{U(L_o) \times U(L_u)}$$

  a point on a Grassmanian manifold (symmetric space!)
- The ground state can be viewed as a map from T_d to the Grassmanian



- Question: how many topologically distinct ground states are there?
- Look at the homotopy group (Kitaev)

$$\pi_{T_d}(\text{Grassmanian}),$$

in general, $\pi_{T_d}(\text{Symmetric Space})$

Topology of Ground States

- Calculation of homotopy groups is hard! Remarkable simplification occurs when L is "large"

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$

(Wikipedia)

- Homotopy groups for large L are familiar Abelian groups ($\mathbb{Z}, \mathbb{Z}_2$)

...leads to

Periodic Table

Cartan\ d	0	1	2	3	4	5	6	7	8	
Complex case:										
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
Real case:										
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\cdots$
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$\cdots$
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

(Kitaev (2009), Ryu et. al. (2010), Ludwig:1512:08882)

Key features

- In any d there are 5 classes that host topologically distinct states
- Bott periodicity: The table has a periodicity of 2 for the "complex" classes, and a periodicity of 8 for "real" classes
- The "nontrivial" classes in $d+1$ dimension are related to those in d
- Nontrivial topology will reflect in properties, gapless surface states etc...

Periodic Table...for "normal" human beings

Simple illustration of the idea in $d = 0$ with $L = 2$ with a single fermion

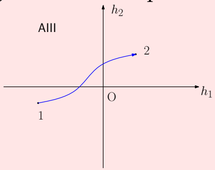
Cartan\d	0
Complex case:	
A	$\mathbb{Z}$
AIII	0
Real case:	
AI	$\mathbb{Z}$
BDI	$\mathbb{Z}_2$
D	$\mathbb{Z}_2$
DIII	0
AII	$2\mathbb{Z}$
CII	0
C	0
CI	0

Class AIII

- Hamiltonian

$$\mathbf{H} = \begin{pmatrix} 0 & h_1 + \mathrm{i}h_2 \\ h_1 - \mathrm{i}h_2 & 0 \end{pmatrix}$$

- Eigenvalues $\pm\sqrt{h_1^2 + h_2^2}$, negative state occupied



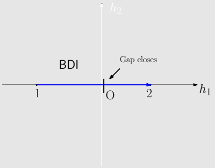
- Can deform any typical system 1 to 2 *without* closing the gap...topologically trivial

Class BDI

- Hamiltonian

$$\mathbf{H} = \begin{pmatrix} 0 & h_1 \\ h_1 & 0 \end{pmatrix}$$

- Eigenvalues $\pm|h_1|$, negative state occupied



- *Cannot* deform system 1 to 2 without closing the gap...two distinct "topologies" described by a "parity" $\mathbb{Z}_2$!