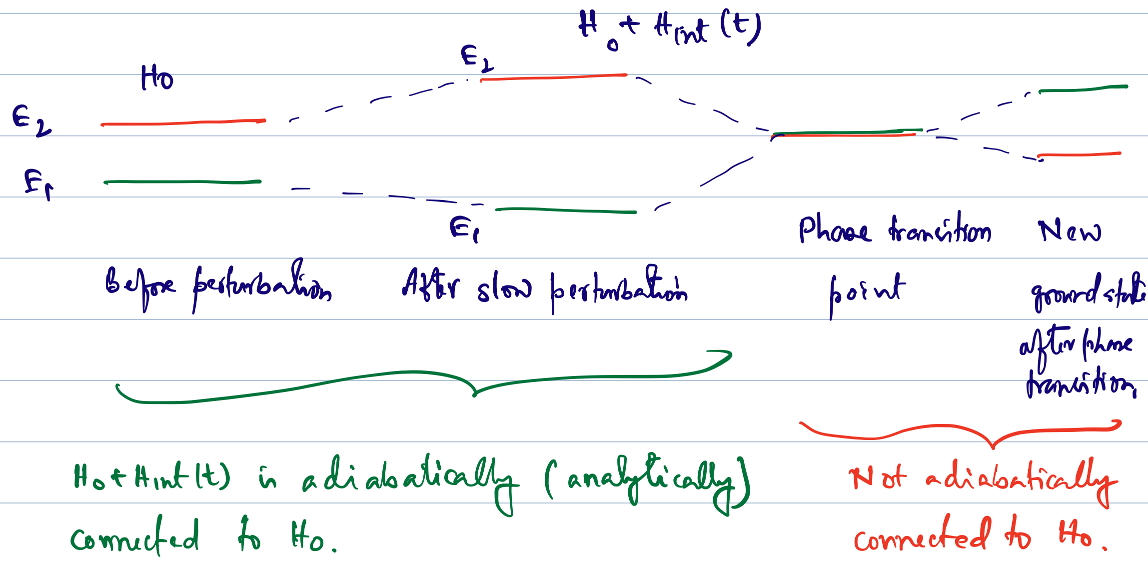
# Chapter 5: Fermi-Liquid Theory

Ref: E. Fradkin's Lecture notes (Shared on Teams)
P. Colemann Book.
G. Vignale Book.

The Fermi-liquid theory, developed by L. Landau in 1957, describes the low-temperature behavior of a metal under interaction. The key message of the Fermi liquid theory is that if the interaction is only density-density interaction, then the low-energy bands of the interacting electrons ($E_k$) have one-to-one correspondence to the non-interacting dispersions ($\epsilon_k$) near the Fermi-levels. Then the interaction term only modifies the band velocity $v_k$ and band mass $m_k$, however, each k-states of the non-interacting electrons corresponds to the same k-state of the interacting electrons (called quasiparticles). Since k is the quantum number here, so, the energy levels (bands) before and after the interactions correspond uniquely to each other. This is the fundamental assumption of the adiabatic theory, that we have learned in quantum mechanics-II course.

- According to the adiabatic theory, we start with, say, two energy levels before the perturbation is turned on. Then, the interaction (which is the perturbation here) is turned on slowly enough that, at each time, we have solved a time-independent Schrödinger equation instantaneously, and that the energy eigenvalues are "continuously" (i.e. "adiabatically") connected to those of the non-interacting case. Therefore, the quantum number, Hilbert space dimension remain the same, and each eigenstate maintain its quantum number. This means, each eigenvalues are shifted up or down by interaction, but the energy gap between them has <u>not closed</u>. On the contrary, if the energy gap between any two energy levels close by the perturbation, the ground state has changed, and one has a phase transition at the gap closing point. Gap closing and phase transition means singular behavior - which violates adiabatic continuity theorem. Therefore, adiabatic continuity is nothing but an analytical continuity in the eigenfunction space. Pictorically:




- Now we want to extend this adiabatic theory to the fermionic band structure $\varepsilon_k$ under a density-density interaction.

  At the outset, let us point out where it won't work:

  (i) Low energy / Low temperature Physics:

  The adiabatic continuity theory is only claimed for the low-energy bands, ie, bands near the Fermi-level. We won't be claiming an adiabatic continuity to states far away from the Fermi level (see below for the reasoning).

  (ii) No phase transition and/or band gap opening

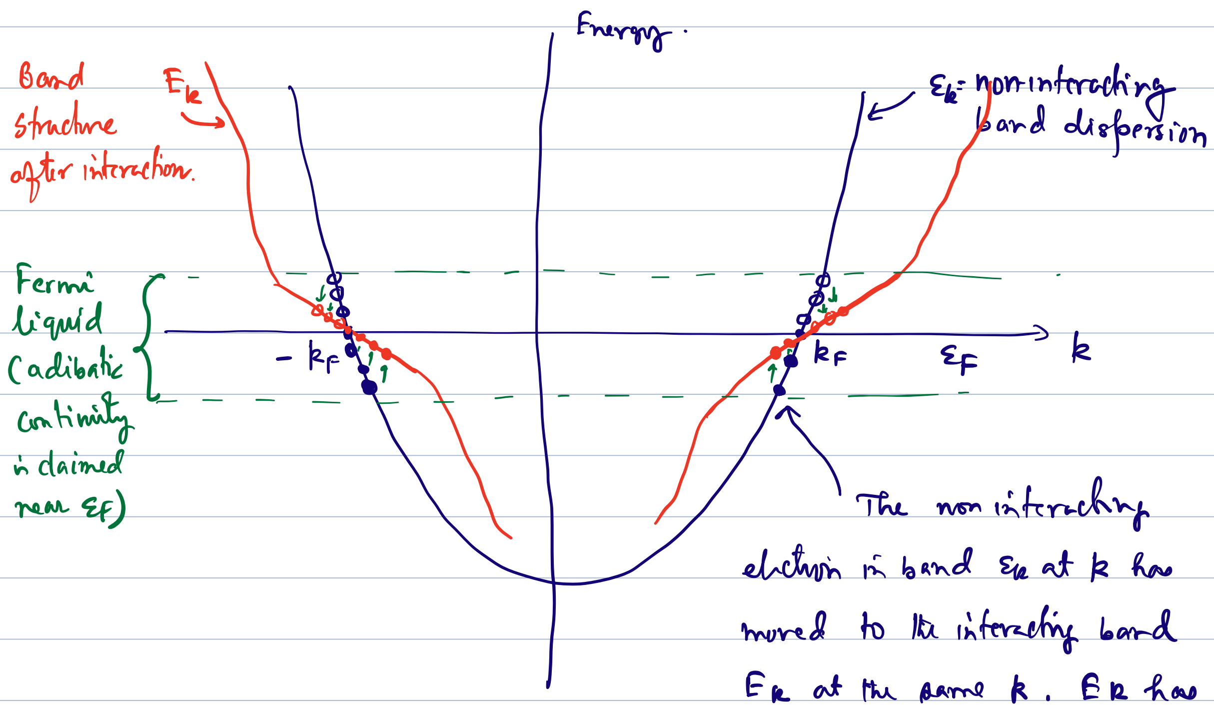
  The theory only holds for metals, but also not for all metals. Because, we do not want any phase transition in the system, which will prohibit the adiabatic continuity to the non-interacting (tight binding) band structure.

(iii) Only density-density interaction:

The Fermi-liquid theory is only proven to hold for density-density interaction $H_{int} \sim v_{kk'} n_k n_{k'}$, not known whether it works for any other interaction. This is, because, the theory is based on the momentum density '$\hat{n}_k$' being roughly a conserved quantity at each $k$ close to $k_F$ with out and with interaction.

## ⊛ Renormalized Band dispersion

We now elaborate on these assumptions and the physical picture before formalizing it into computations of thermodynamic and transport properties.




The non interacting electron in band $\varepsilon_k$ at $k$ has moved to the interacting band $E_k$ at the same $k$. $E_k$ has lower energy (compare to $\varepsilon_F$) than $\varepsilon_k$, i.e., less velocity and higher mass than free electron velocity and mass.

Based on the above observation, we can roughly say the interacting band structure is proportional to the non-interaction one by a factor $Z < 1$, near the fermi level, as

$$E_k = Z\,\varepsilon_k \qquad -- (1)$$

[→ We have readjusted the chemical potential in the interaction case to match the fermi-momentum $k_F$ same in both cases. This is for simplicity in discussion.

→ We have also assumed the factor $Z$ to be $k$-independent. This turns out to be the case for many isotropic Fermi surfaces and onsite interaction, but can be generalized to $Z_k$.]

Now writing $E_k = v_F(k-k_F)$ and $\varepsilon_k = v_F^0(k-k_F)$, we see that the band velocity of the interacting electrons has reduced as

$$v_F = Z\,v_F^0 \quad \text{where } Z<1. \qquad [\hbar = 1].$$

Similarly the effective mass of the interacting electrons has increased by $Z^{-1}$: $m^* = k_F/v_F = Z^{-1} k_F/v_F^0 = Z^{-1} m_0^*$.

$Z$ is called the quasiparticle residue, or, the renormalization factor. The justification of this name comes from the Green's function description of the quasiparticle dynamics, which will not be covered here.

Therefore, the interacting bands near the fermi level becomes flatter due to interaction by a change of slope by $Z$. The fermi-liquid theory does not predict the quasiparticle dispersion picture far away from the Fermi-level. In fact, the quasiparticle picture breaks down at high energies.

## ⊛ Renormalized momentum density / momentum distribution function.

The above discussion was about the quasiparticle energy. We will not be discussing about the quasiparticle states, which is simply governed by the time-evolution operator with the quasiparticle energy. This description of wavefunction is not quite adequate as we will learn that the quasiparticles have finite lifetime and modelling finite lifetime is difficult through Hamiltonian description. We need Green's function method to include finite lifetime as we saw a demo for the response function.

The easier thing to look at the momentum density $n_{k,\sigma} = c_{k\sigma}^{+} c_{k\sigma}$.
Because in the noninteracting Hamiltonian $H_0 = \sum_{k,\sigma} \epsilon_k n_{k\sigma}$, $n_{k\sigma}$ are conserved quantities, and one has N number of conserved quantities for a N-dimensional Hilbert space (N = # k points × 2 spin). So it's an integrable system. Then at T=0, one has $\langle n_{k,\sigma}\rangle_0 = \theta(k_F - k)$, satisfying the Fermi statistics.

Landau's argument was that the density-density interaction produces virtual excitations which are particle-hole pairs. So the momentum density $n_{k\sigma}$ adiabatically evolve to a value $\langle n_{k,\sigma}\rangle_{int}$ exactly at the same momentum k, and spin σ, and has a discontinuity at $k = k_F$, but the discontinuity value is reduced by Z as

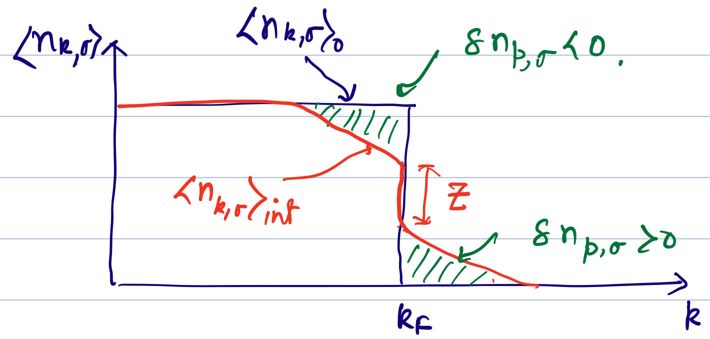
$$\langle n_{k,\sigma}\rangle_{int} = Z\,\theta(k_F - k) \quad ---\cdot (2).$$

This gives Z less number of electrons inside the fermi momentum $k_F$. So, now one has

$$\delta n_{p,\sigma} = \langle n_{p,\sigma}\rangle_{int} - \langle n_{p,\sigma}\rangle_0$$
$$= 1 - Z, \text{ at } T = 0.$$

(We change the index k to p to emphasize that p is defined w.r.to $k_F$.)

amount of electrons pulled out from the non-interacting Fermi sea and filled them above the Fermi-level (see the figure below). This is roughly the "ground state" of the interacting Hamiltonian - which is made of the low-energy particle-hole excitations of the non-interacting Hamiltonian. And we only have one parameter to describe them. and that is $\delta n_{p,\sigma}$ (or $Z$ at $T=0$).




Some caveats to remember. Denoting it as the "ground state" of the interacting Hamiltonian $H_0 + H_{int}$ is not quite accurate because these states with $\delta n_{k,\sigma}$ particle-hole excitations are not an eigenstate of $H_0 + H_{int}$. Rather one should think of it as some approximate (variational) ground state which is adiabatically (analytically) connected to the non-interacting Fermi sea ground state $|FS\rangle$. Here $\delta n_{p,\sigma}$ is the variational parameter which should be obtained self-consistently.

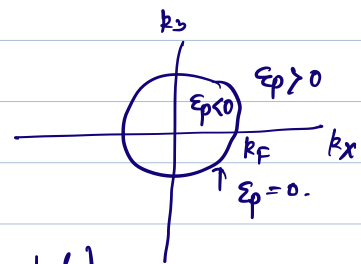
Let us denote this excitation density by creation and annihilation operators $C^{\dagger}_{p\sigma}, C_{p\sigma}$ as

$$\delta n_{p,\sigma} = C^{\dagger}_{p\sigma} C_{p\sigma}. \quad --(3).$$

But now you have to be careful in interpreting these creation/

annihilation operators. These are the "excitations" that are created from the non-interacting Fermi sea. This means, the Fermi-sea is the vaccum state for these excitations, and these excitations are called "quasiparticles" with reduced velocity $Z v_k$ and enhanced mass $Z^{-1} m^*$, compared to non-interacting electron's velocity $v_k$ and mass $m^*$.

So, the number of excitations one creates from the Fermi sea is not really a conserved quantity - they however have finite lifetime which decreases as we move away from the fermi level that we will see later. Therefore, this interacting ground state is like a grand cannonical ensemble in which the number of quasiparticles is not conserved. So, we can freely add or remove the quasiparticles to the interacting system such that we obtain a lowest energy many body state. We add a quasiparticle to the ground state at $p > 0$ (since $p \equiv k - k_F$) with quasiparticle energy $\varepsilon_p > 0$, and the corresponding state is $c_p^+ |FS\rangle$. Similarly,




we remove a quasiparticle (or create a quasihole) in the Fermi sea at $p < 0$ with quasihole energy $\varepsilon_p < 0$ and the corresponding state is $c_p |FS\rangle$. Since the change of number density due to removal of electron is $\delta n_{p,\sigma} < 0$, so, the effective excitation energy is $(-\varepsilon_p)(-\delta n_{p,\sigma}) = \varepsilon_p \delta n_{p,\sigma} = E_p > 0$. Therefore, adding both quasiparticles and quasiholes above and below the Fermi level costs the same energy $|E_p|$. It costs no energy to add a quasiparticle or quasihole at the Fermi momentum $E_{p_F} = 0$.

→ The ground state of the Fermi-liquid can hence be thought as

$$|G_{FL}\rangle = \prod_{p < p_F} \prod_{p' > p_F} c_{p'}^+ c_p \, |FS\rangle .$$

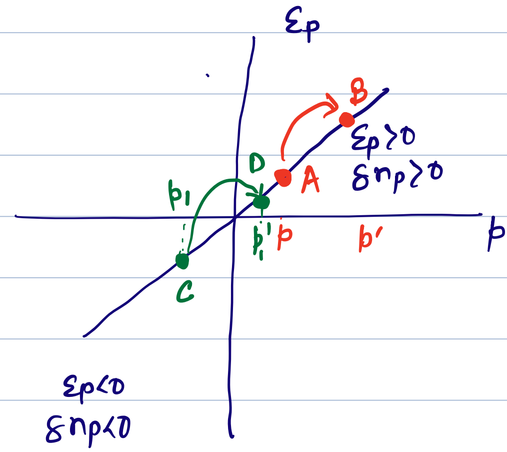
because we create (destroy) a quasiparticle (quasi hole) above (below) the Fermi level.

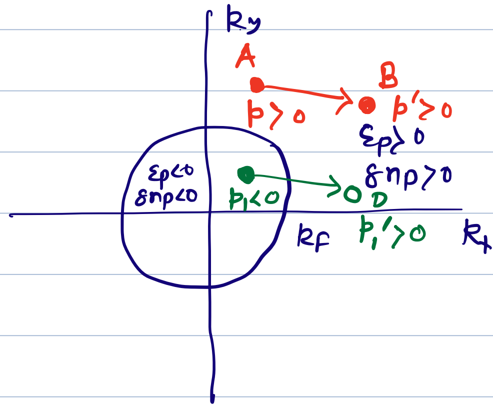
## (*) Interaction & life time :

We have not yet talked about the effect of the interaction in any meaningful way. These quasiparticle density operators $\delta n_{p,\sigma}$ continue to commute with the kinetic energy term and hence under the time-evolution of the K-E., they do not decay. But the $\delta n_{p,\sigma}$ do not commute with the interaction term (not even with the density-density interaction). So, if we evaluate the scattering matrix element of the interaction term, $\langle p_1' p_2' | H_{int} | p_1 p_2 \rangle$, we will have off-diagonal terms. This scattering process gives a finite life time of the quasiparticles to be in the same state $|p\rangle$ before it scatters off to some other state $|p'\rangle$ due to interaction. Much like the Drude's semiclassical picture of scattering relaxation time $\tau$ for each quasiparticle (and quasi holes) which is the average lifetime of a quasiparticle in a given state $|p\rangle$ before it scatters off to another state $|p'\rangle$.

Landau gave a qualitative argument on why the quasiparticle lifetime $\tau$ becomes infinite as we approach the Fermi surface, and it varies as $\tau \sim 1/\varepsilon_p^2$, where $\varepsilon_p$ is the quasiparticle energy measured with respect to the Fermi energy, so that $\varepsilon_p = 0$ at $p = 0$ (at the Fermi level) and that $\tau \to \infty$ at the Fermi level. This argument can be given qualitatively based on the phase space argument for the scattering process across a Fermi surface. The derivation will be presented later, now we discuss the argument.

The quasiparticle dispersion and the Fermi surface are qualitatively shown below.



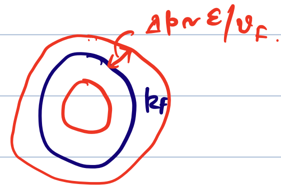




Suppose we add a quasiparticle at A, at a momenta $p>0$, above the Fermi-level. Under the interaction, this quasiparticle can only scatter to another momentum $p'>0$ which is also above the Fermi level (say at B). The energy released during this scattering is $E_B - E_A$, and momentum released is $p'-p$. This energy will now be absorbed by another quasiparticle which was sitting below the Fermi level, say at C at $p_1 < 0$ momentum. This quasiparticle will now be moved to above the Fermi level only, say to D, at $p_1' > 0$ such that the energy and momentum are conserved as $E_B - E_D = E_D - E_C$ and $p' - p = p_1' - p_1$. Using a Born approximation, one can estimate the life time as

$$\frac{1}{\tau} \propto \left| \langle p_1' p \,|\, H_{int} \,|\, p\, p_1 \rangle \right|^2 \sim \varepsilon^2 . \qquad \text{---}(4)$$

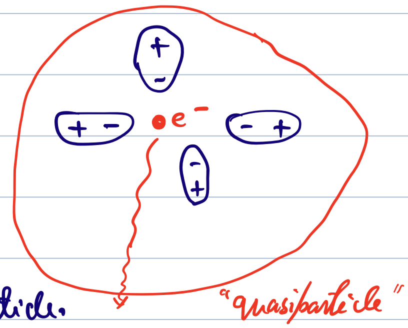
→ We immediately see that there is no scattering at zero energy because, that amounts to quasiparticles on the Fermi surface scatter to another momenta on the Fermi surface – which is not possible since all states on the Fermi surface is already occupied. This blocking of the scattering on the Fermi surface is entirely due to the Pauli exclusion principle (called Pauli blocking) and is valid irrespective of any interaction that one choses. Therefore, the quasiparticles on the Fermi surface has infinite lifetime - irrespective of the interaction

→ Now, for an scattering at energy $\varepsilon > 0$, the scattering is restricted within the momentum shell of $\Delta p \sim \varepsilon / v_F$ above the Fermi level and same below the Fermi level. One cannot scatter any quasiparticle outside this shell, because the quasiparticle inside the Fermi surface need to come out of the Fermi surface. This restriction on the available phase space for the scattering at a given energy puts the restriction on the inverse lifetime of the quasiparticles residing within this momentum shell.




→ For another quasiparticle at a higher momentum, the available phase space for scattering is also higher. Now since the scattering due to interaction is a two-body interaction, so, the scattering crosssection is a joint probability of two scattering processes of equal phase space, each phase space is proportional to $\varepsilon \sim v_F \Delta p$. Therefore, the scattering crosssection or inverse liftime is proportional to $\varepsilon^2$:
$\tau^{-1} \sim \varepsilon^2$.

• Another physical picture of quasiparticle description is that as an electron moves in a metal, it repels some of the electrons around it, and thereby it creates polarization cloud - called the vacuum polarization. The entire charge cloud arond the electron is the effective particle here - which is called the quasiparticle.




So, the word quasi only signifies that its an electron with a charge cloud around it. however, its charge, spin, wave vector remains the same to the original electron, and its velocity (mass) are decreased (increased) due to the charge cloud.

But the vacuum here is the Fermi sea, and the quasiparticles are created by scattering an electron near the Fermi surface. A small momentum transfer around the Fermi wave vector is like a very slow or long-distance change of wavelength with respect to the Fermi wave length wavelength. Scattering of such large volume quasiparticle is hence discouraged and hence these quasiparticles survive longer.

⊛ Therefore, the bottomline is that the quasiparticles are very resilient to the interactions and are long-lived near the Fermi-level - purely due to their quantum statistics, and irrespective of any interactions. This is the reason, the quasiparticle picture - which is based on adiabatic / analytic continuity to non-interaction / free electron behavior - as postulated with in the Fermi Liquid theory works in correlated metal with a phase transition.

## 5.2 Computations of Response functions in the Fermi liquid Theory

With the specific analytical continuation of an interacting problem to an non-interacting one as defined in eq(1), we can infact compute / estimate the renormalization of various response functions that we measure experimentally in terms of the single renormalization parameter $Z$. This does not require to assume any Hamiltonian or any specific form. We first discuss this before deriving the low-energy Hamiltonian Landau predicted.

There are a few typical response functions that we often measure and compute, which are electron's spectral weight, specific heat, charge, spin susceptibilities, electrical conductivity, etc.

⊛ The Fermi velocity changes by $v_F = Z v_F^0$. Therefore, the velocity of the quasiparticle decreases since $Z<1$.

→ The effective mass changes by $m^* = Z^{-1} m$.

We measure effective mass via Shubnikov-de Haas van Alphen (SdHvA) oscillation from the slope of the oscillation frequency versus FS area.

$Z^{-1} = m^*/m \sim 2.8$ in $He^3$, and it can be 100-1000 in heavy-fermion metals.

→ We can linearize the quasiparticle band near $p_F$ as

$$\varepsilon_p \approx \hbar v_F (p - p_F) \approx Z \varepsilon_p^0 \quad \text{--- (8a)}$$

$$\approx \frac{\hbar p_F}{m^*} (p - p_F) \quad \text{--- (8b)}$$

$v_F$ can be measured directly from the angle (momentum) resolved photoemission spectroscopy (ARPES) which measures quasiparticle dispersion.

→ Fermi energy ($\varepsilon_F$): We said the shift of the fermi wavevector ($p_F$) is trivial, but change in fermi energy is $\varepsilon_F = \hbar^2 p_F^2 / 2m^* = Z \varepsilon_F^0$. So, Fermi energy decreases by Landau parameter. So, does the Fermi temperature $T_F = \varepsilon_F / k_B = Z T_F^0$.

→ The quasiparticle density of states $d(\varepsilon)$ is defined as

$$d(\varepsilon) = \frac{1}{V} \sum_{p,\sigma} \delta(\varepsilon - \varepsilon_{p,\sigma}) = \underset{\text{for spin}}{2} \int \frac{d^3 p}{(2\pi\hbar)^3} \delta(\varepsilon - \varepsilon_p)$$

$$d(0) = 2 \cdot 4\pi \int_0^{p_F} \frac{p^2 dp}{(2\pi\hbar)^3} \left( \frac{\partial p}{\partial \varepsilon_p} \right) \delta(p) = \frac{p_F^2}{\pi^2 \hbar^3} \left. \frac{\partial p}{\partial \varepsilon_p} \right|_{p_F}$$

$$= \frac{m^* p_F}{\pi^2 \hbar^2} = \frac{m^*}{m} d_0(0) = Z^{-1} d_0(0) \quad \text{--- (9)}$$

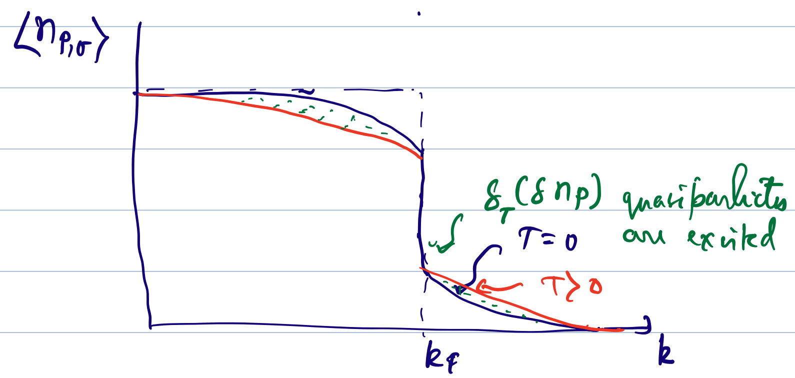
where $d_0(0)$ is the DOS of noninteracting electrons. The conclusion is as the bands become renormalized by $Z$, i.e., its slope decreases by $Z$, the density of states increases by $Z$.

The quasiparticle density of states we measure via, for example, specific heat measurement, through the Sommerfeld coefficient ($\gamma$).

(*) $C_V$ - specific heat is a response function to temperature change and measures the energy fluctuation. The linear response theory of specific heat is

$$\delta E = C_V\, \delta T .$$

During the thermal fluctuation, the quasiparticles are excited across the fermi-level which is captured by the Fermi-Dirac distribution function. For the quasiparticles case, the change in momentum density / momentum distribution function look like




(Assuming $Z$ does not change with temparature.)

→ Interestingly, the temperature dependence of the specific heat does not change by the Landau parameters, only its prefactor, ie, the Sommerfeld coefficient which depends on the density of states, becomes renormalized. This can be computed easily - with a hand waving argument or more rigorously. Let me give the hand-waving argument here, where the rigorous calculation is given in the homework.

Specific heat measures the energy fluctuation due to temperature. At a given temperature T, the number of states that are excited across the fermi-energy is $\delta n_p$. Now, on a spherical (or equivalent) Fermi surface, we only excite the particles radially outward, because the angular directions are degenerate (equi-energy surface). Therefore, irrespective of any dimension of the Fermi-surface, only the radial dimension is active for thermal excitation. Therefore, according to equipartition argument, the number of modes thermally excited in $\delta n_{p,\sigma} \propto k_B T$. Each mode carry quasiparticle energy is $\sim k_B T$. Therefore, the total internal energy fluctuation is

$$E \sim \sum_{p,\sigma} \delta n_{p,\sigma}\, \varepsilon_p = \frac{\pi^2}{3} d(0)\, k_B^2 T^2 = \frac{\pi^2}{3} Z\, d_0(0)\, k_B^2 T^2 \quad \text{--(10)}$$

(This temperature exponent is independent of the dimension of the material. This is an interesting manifestation of the Fermi statistics which does not hold for bosons.)

Also, we notice that the information about the Hamiltonian is only in the density of states

Then the specific heat is given by

$$C_V = \frac{\partial E}{\partial T} = \frac{\pi^2}{3} d_0(0)\, k_B^2 T = \gamma T \quad \text{--(11)}$$

Sommerfeld coefficient

Therefore, the Sommerfeld coefficient is renormarized (enhanced) b compared to the free electron system by $Z^{-1}$:

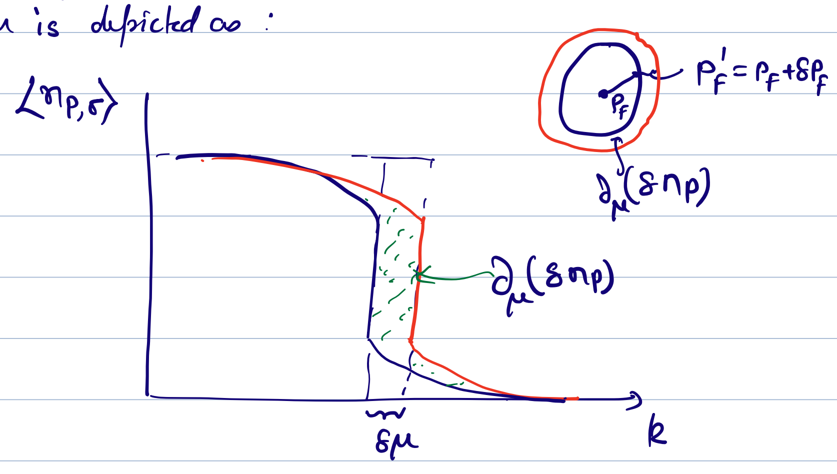
$$\boxed{\gamma = Z^{-1}\gamma_0} \quad \text{--- (12).}$$

$\chi_c$ – Charge susceptibility or compressibility. This is a response function to electrostatic potential, which causes density fluctuations. We are here looking at $\chi_c(q=0, \omega=0)$, ie, the global (spatially uniform) electrostatic potential - which is the change in chemical potential $\delta\mu$, and measuring the change in number of electrons $\delta n_c$, which are related to each other by the linear response theory as

$$\delta n_c = \chi_c \, \phi_{ext} \Rightarrow \delta n_c = -\chi_c \, \delta\mu \Rightarrow \boxed{\chi_c = -\frac{\partial n_c}{\partial \mu}}$$

$$\delta n_c = \frac{1}{2}(\delta n_\uparrow + \delta n_\downarrow)$$

- Pictorially, the quasiparticle density change due to change in $\mu$ is depicted as :



The quasiparticle picture is also the free electron picture here in which the chemical potential has changed by $\delta\mu$ due to external electrostatic potential. So, we have

$$H = \sum_{p} \varepsilon_{p\sigma} c^{\dagger}_{p\sigma} c_{p\sigma} \xrightarrow{\delta\mu} \sum_{p,\sigma} (\varepsilon_{p\sigma} - \delta\mu) c^{\dagger}_{p\sigma} c_{p\sigma}.$$

The change in number of quasiparticles is

$$\partial_{\mu}(\delta n_{c,p}) = \frac{\partial}{\partial\mu} \frac{1}{2} \sum_{p,\sigma} \langle \underbrace{c^{\dagger}_{p\sigma} c_{p\sigma}}_{f(\varepsilon_p)} \rangle$$

$$= \frac{1}{2} \sum_{p,\sigma} \underbrace{\frac{\partial}{\partial\mu} f(\varepsilon_{p,\sigma} - \delta\mu)}_{\delta(\varepsilon_{p,\sigma}) \text{ at } T \to 0.}$$

$$= -\frac{1}{2} \sum_{p,\sigma} \delta(\varepsilon_{p,\sigma})$$

$$= -d(0)$$

Therefore, $\boxed{\chi_c = -d(0) = -Z^{-1} d^{0}(0) = Z^{-1} \chi_c^{0}}$

i.e., the charge susceptibility is enhanced by the interaction

- The compressibility $\kappa$ is defined as $\kappa = \frac{1}{V} \frac{\partial P}{\partial V}$. Now $n = N/V$, and hence $\kappa = n^2 \frac{\partial \mu}{\partial n} = n^2 \frac{1}{\chi_c}$.

So, the electronic contribution to the compressibility is related to the charge fluctuation, i.e., isotropic FS deformation.

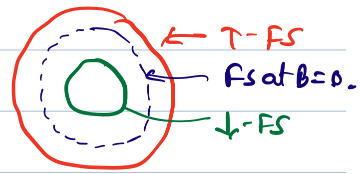
## Magnetic susceptibility :

→ $\chi_s$ – Spin susceptibility – is very similar to the charge susceptibility, and is a measure of spin density fluctuation $\delta n_s$ due to external magnetic field B (spin density is related magnetization by the Bohr magneton $\mu_B$).

So, using linear response theory we have :

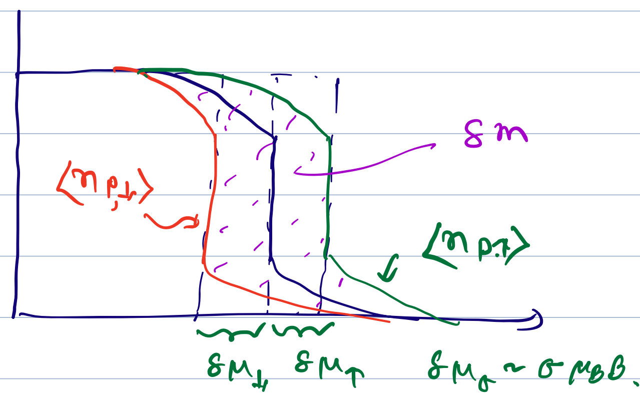
$$\delta m = \mu_B \, \delta n_s = \chi_s B$$

For this case, the quasiparticles will move from ↓-spin state to ↑-spin state, hence the non-interacting FS will split into two → ↓-spin FS will shrink while ↑-spin FS expands.




- The two spin density $\delta n_s$ interact via the $F^{(A)}_{pp'}$ component, and one expect the leading term to be $F^{(A)}_{\ell=0}$ Landau parameter.

- The pictorial view of the change in the distribution function in



In a paramagnet, which is a spin degenerate system, the magnetic susceptibility $\chi(q=0, \omega=0)$ is same as the charge susceptibility. We obtained the charge susceptibility in the previous chapter due to density - density correlation within the linear response theory. We found the result for uniform and $\omega \to 0$ case to be as $(\chi_{spin} = \mu_B^2 \chi_{charge})$

$$\chi(0,0) \sim \mu_B^2 \frac{\partial n}{\partial \mu} = \text{Compressibility}$$

$$= \mu_B^2 \, d(0) \quad = \text{Density of states at } \mu.$$

$$\cdots (13)$$

→ Here again we can derive it in a hand-waving manner, while more rigorous calculation is saved for the home work. A magnetic field $B$ splits the Fermi surface for $\sigma = \uparrow$ & $\downarrow$ spins, as the number of quasiparticles in $\uparrow$ & $\downarrow$ states are now different. The quasiparticle energies for up & down spins are $\quad \varepsilon_{p,\sigma} = \varepsilon_p - \sigma \mu_B B$, where $\sigma = \pm 1$ for $\uparrow/\downarrow$ states

We can absorb the magnetic energy into spin resolved chemical potential $\mu_\sigma = \mu + \sigma \mu_B B$. This shrink / expands the Fermi surfaces for $\sigma = \downarrow$ & $\uparrow$ (by equal amount) and the corresponding change in quasiparticle number is

$$\delta N_\sigma = (\text{density of states / per energy unit}) \times (\text{energy difference})$$

$$= d(0)\,(\sigma \mu_B B)$$

The magnetization $M = \frac{1}{2}\mu_B (N_\uparrow - N_\downarrow)$

$$= d(0)\, \mu_B^2 B.$$

Then using the linear response theory, we obtain the spin susceptibility $\chi$ as

$$\chi_s = M/B$$
$$= \mu_B^2 d(0) \quad \text{as in eq (5)}$$

Here also we can derive the result for quasiparticle dispersion and it can be obtained that $\chi(0,\delta)$ has the same form as

$$\chi_s = \mu_B^2 d(0)$$
$$= \mu_B^2 Z^{-1} d_0(0)$$
$$= Z^{-1} \chi_s^0, \text{ where } \chi_0 = \text{non-interacting susceptibility}$$

Therefore, the magnetic susceptibility is also enhanced by the same factor $Z^{-1}$.

→ Wilson Ratio: Wilson ratio (also called the Stoner enhancement factor) is defined as

$$W = \frac{\chi_s}{\gamma} = \frac{\mu_B^2 d(0)}{\frac{\pi^2}{3} d(0) k_B^2} = 3\left(\frac{\mu_B}{\pi k_B}\right)^2 . \quad \cdots (14)$$

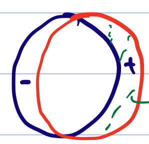
remains unchanged as both $\chi$ and $\gamma$ are enhanced by the same renormalization factor $Z^{-1}$.

But it turns out that a careful calculation of self-energy and renormalization factor $Z$ gives slightly different values of $Z$ for $\gamma$ & $\chi$ due to different sort of response and different mechanism of fluctuation we are computing here. We will see that in terms of the Landau parameters.

(*) $\sigma$ - Electrical conductivity is a current density - current density response function due to applied electric field $\bar{E}$. According to Ohm's law we have

$$J = \sigma E.$$

Microscopically, electric field boosts the electron's velocity along the direction of the field, and hence the electron's momentum change $\vec{p} \to \vec{p} - \frac{e}{c}\vec{A}$. This shifts the Fermi surface along the field direction




$\vec{J} \sim e(\delta n_c)\,\vec{v}$, $\vec{v} = \frac{\vec{p}}{m}$

This perturbation does not cause any change in density, but only boosts them along the field direction. Since $\delta n_c$ interact with other $\delta n_c$ by $F^{(s)}_{p,p'}$, their velocity is slower. The renormalization of velocity due to $F^{(s)}_{p,p}$ will have leading order correction from $F^{(s)}_{\ell=1}$, i.e., from $\ell=1$ angular momentum term, not from $\ell=0$ term.

**Table. 8.1 Key Properties of the Fermi Liquid .**

| PROPERTY | NON-INTERACTING | LANDAU FERMI LIQUID |
|---|---|---|
| Fermi momentum | $p_F$ | unchanged |
| Density of particles | $2\frac{V_{FS}}{(2\pi)^3}$ | unchanged |
| Density of states | $N(0) = \frac{mp_F}{\pi^2\hbar^3}$ | $N^*(0) = \frac{m^* p_F}{\pi^2\hbar^3}$ |
| Effective mass | $m$ | $m^* = m(1 + F_1^s)$ |
| Specific heat Coefficient $C_V = \gamma T$ | $\gamma = \frac{\pi^2}{3}k_B^2 N(0)$ | $\gamma = \frac{\pi^2}{3}k_B^2 N^*(0)$ |
| Spin susceptibility | $\chi_s = \mu_F^2 N(0)$ | $\chi_s = \mu_F^2 \frac{N^*(0)}{1+F_0^a}$ |
| Charge Susceptibility | $\chi_C = N(0)$ | $\chi_C = \frac{N^*(0)}{1+F_0^s}$ |
| Collective modes | - | Sound ($\omega\tau << 1$)<br>Zero sound ($\omega\tau >> 1$) |

← $d(0)$ in our notation

$Z^{-1} = (1 + F_1^{(s)})$

← Notice it's not $F_1^{(s)}$, but $F_0^{(a)}$

$Z^{-1} = 1 + F_0^{(S,A)}$

Table 8.1 summarizes the key properties of the Landau Fermi liquid.

### 7.3.1 Landau Parameters

The power of the Landau Fermi liquid theory lies in its ability to parameterize the interactions in terms of a small number of multipole parameters called "Landau Parameters". These parameters describe how the original non-interacting Fermi liquid theory is renormalized by the feedback effect of interactions on quasiparticle energies.

In a Landau Fermi liquid in which spin is conserved, the interaction is invariant under spin

## 5.3. The effective Hamiltonian of the quasiparticles

The picture we gather above is that we start with non-interacting electrons occupying a Fermi surface. Then the action of turning on the interaction (slowly) is that it excites a few electrons ($\delta n_{p,\sigma}$) across Fermi level. These excited electrons go through the interaction, and their dynamics is governed by an effective low-energy Hamiltonian $H$ which is a functional of $n_{p,\sigma}$: $H(n_{p,\sigma})$. Moreover, we assume that the energy $E(n_{p,\sigma})$ is an analytic functional of $n_{p,\sigma}$ such that we can perform an Taylor expansion around their non-interacting value of $n^0_{p,\sigma}$ as

$$E(n_{p,\sigma}) = E_0 + \sum_{p,\sigma} \underbrace{\left.\frac{\delta E}{\delta n_{p,\sigma}}\right|_{n^0_{p,\sigma}}}_{\epsilon^0_{p,\sigma}} \underbrace{(n_{p,\sigma} - n^0_{p,\sigma})}_{\delta n_{p,\sigma}} + \frac{1}{2}\sum_{\substack{p,p'\\ \sigma,\sigma'}} \underbrace{\left.\frac{\delta^2 E}{\delta n_{p\sigma}\,\delta n_{p'\sigma'}}\right|_{n^0_{p,\sigma} n^0_{p'\sigma'}}}_{f^{\sigma\sigma'}_{pp'}} \delta n_{p\sigma}\,\delta n_{p'\sigma'} + O(\delta n^3) \quad \text{--- (5)}$$

- The first term is the kinetic energy term for the quasiparticles density $\delta n_{p,\sigma}$ without the interaction, defined by, $\varepsilon^0_{p,\sigma} = \left.\frac{\delta E}{\delta n_{p,\sigma}}\right|_{n^0_{p,\sigma}}$

- $f^{\sigma\sigma'}_{p,p'} = f^{\sigma'\sigma}_{p'p}$ is called the Landau parameter. This is the only parameter in the theory. This describes interaction between quasiparticles near the Fermi surface, and evaluated at $\delta n_{p,\sigma} = 0$, where all quasiparticles are "frozen" or inside the Fermi sea.

- The dispersion of the interacting quasiparticle (the one that we measure) is

$$\boxed{\varepsilon_{p,\sigma} = \frac{\delta E}{\delta n_{p,\sigma}} = \varepsilon^0_{p,\sigma} + \sum_{\sigma' p'} f^{\sigma\sigma'}_{pp'}\,\delta n_{p'\sigma'} \quad \text{--- (6)}}$$

→ The second term is exactly the self-energy correction (to non-interacting electrons):

$$\Sigma_{p,\sigma} = \sum_{p'\sigma'} f_{pp'}^{\sigma\sigma'} \delta n_{p'\sigma'} \quad \text{--- (7)}$$

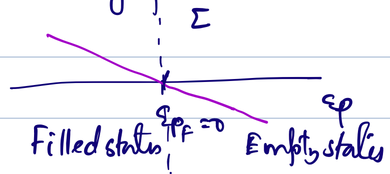
due to the interaction, which is defined in the Green's function technique. In the Green's function technique, one only includes the time-dependence and obtain a frequency dependent term, and the imaginary part of the self-energy gives the life time of the quasiparticle (energy dissipation in the spectral weight representation). Here the imaginary part is not captured. The self energy method is general and obtained for other interactions, not only the Landau parameter.

→ Then the renormalization factor $Z$ is obtained from the frequency derivative of the real part of the self-energy. Here within the time-independent case, frequency $\omega = \epsilon_{p,\sigma}$ and hence the renormalization factor is obtained as

$$\boxed{Z = 1 + \left.\frac{\partial \Sigma_{p,\sigma}}{\partial \epsilon_{p,\sigma}}\right|_{p_F}} \quad \text{--- (8)}$$

→ This is again the general definition for the renormalization factor. We will learn below to compute it for the Landau interaction case.

→ Since $Z \leq 1$, and it's defined by the slope of $\Sigma$ near $p_F$, so, we can roughly predict




[Using the Kramers Kronig relation, we can say $\Sigma''$ (inverse lifetime) $\to 0$ as $\epsilon \to 0$.]

- This is the low-energy dispersion of the quasiparticles, which differ from its non-interacting (tight binding / parabolic) dispersion $\varepsilon^{0}_{p,\sigma}$ by the last term. The last term incorporates summation over all possible interaction with the matrix-element $f^{\sigma\sigma'}_{pp'}$. Before making further manipulation and approximation on $f^{\sigma\sigma'}_{pp'}$, let us first look at a few definitions such as $v_F$, $m^*$, $d(\varepsilon_F)$, $Z$ in terms of $f^{\sigma\sigma'}_{pp'}$.

  We extract a definition of $Z$ from the band dispersion in eq(6) by doing a Taylor expansion of the 2nd term near $\varepsilon^{0}_{p,\sigma}$ as

$$\varepsilon_{p,\sigma} = \varepsilon^{0}_{p,\sigma} + \sum_{p'\sigma'} \left[ f^{\sigma\sigma'}_{pp'} \left.\frac{\partial \delta n_{p'\sigma'}}{\partial \varepsilon_{p\sigma}}\right|_{\varepsilon^{0}_{p,\sigma}} + \left.\frac{\partial f^{\sigma\sigma'}_{pp'}}{\partial \varepsilon_{p,\sigma}}\right|_{\varepsilon^{0}_{p,\sigma}} \right] \varepsilon^{0}_{p,\sigma} .$$

$$= Z\, \varepsilon^{\sigma}_{p,\sigma} \qquad + \mathcal{O}(\varepsilon^{2}_{p\sigma})$$

From here we obtain the first order term in $Z$ as

$$Z = 1 + \sum_{p'\sigma'} \left[ f^{\sigma\sigma'}_{pp'} \left.\frac{\partial \delta n_{p'\sigma'}}{\partial \varepsilon_{p\sigma}}\right|_{\varepsilon^{0}_{p,\sigma}} + \left.\frac{\partial f^{\sigma\sigma'}_{pp'}}{\partial \varepsilon_{p,\sigma}}\right| \delta n_{p'\sigma'} \Big|_{\varepsilon_{p,\sigma}} \right]_{p=p_F} \quad \text{--}(9)$$

→ of course, here we can add a $p$-dependence in $Z$, but we will ignore it, as one would find that $p$-dependence of $Z$ is small near $p_F$.

## 5.4 Landau Parameter (For rotationally invariant systems)

The Landau parameter $f_{pp'}^{\sigma\sigma'}$ is an unknown function. How do we really parameterize a function? It turns out due to spin rotational and spatial rotational symmetries, $f$ only depends on the relative spin orientation and relative momentum orientation between the two quasiparticles. Imposing these symmetries will simplify the form of $f$.

(*) Spin rotational symmetry: Let us first focus on the spin part. In the Landau interaction the total spin $S = S_1 + S_2$ is conserved. Hence we have $f^{\uparrow\uparrow} = f^{\downarrow\downarrow} = f^{(S)}$ (define) and $f^{\uparrow\downarrow} = f^{\downarrow\uparrow} = f^{(A)}$ ($p$-dependence is implied on both sides). $f^{(S)}$ & $f^{(A)}$ are called spin symmetric and antisymmetric (or charge and spin) part of $f$.

Let us define the charge and spin densities as

$$\delta n_c = (\delta n_\uparrow + \delta n_\downarrow)/2 \,, \quad \text{where } p\text{-dependence on} \quad \text{--- (9a)}$$

$$\delta n_s = (\delta n_\uparrow - \delta n_\downarrow)/2 \,. \quad \text{both sides are implied.} \quad \text{--- (9b)}$$

$$\therefore \quad \delta n_\sigma = \delta n_c + \sigma\, \delta n_s \quad , \sigma = \pm 1 \text{ for } \sigma = \uparrow, \downarrow \quad \text{--- (9c)}$$

$$+ \sum_{\sigma\sigma'} \delta n_\sigma \delta n_{\sigma'} = \sum_{\sigma\sigma'} \Big[ \delta n_c \delta n_c + \sigma\sigma' \delta n_s \delta n_s + \underbrace{(\sigma+\sigma')}_{=0 \text{ since } \sigma = \pm 1} \delta n_c \delta n_s \Big]$$

$$= 4\, \delta n_c \delta n_c + 2\, \delta n_s \delta n_s$$

Substituting it in the Landau interaction term we get

$$\boxed{\sum_{\substack{pp' \\ \sigma\sigma'}} f_{pp'}^{\sigma\sigma'} \delta n_{p\sigma} \delta n_{p'\sigma'} = \sum_{pp'} f_{pp'}^{(S)} \delta n_{c,p} \delta n_{c,p'} + f_{pp'}^{(A)} \delta n_{s,p} \delta n_{s,p'}} \quad \text{--- (10)}$$

(we have absorbed factor '4' & '2' in $f^{(S,A)}$).

## ⊛ Spatial rotational symmetry:

→ Since we are interested in quasiparticles near the fermi-surface, so, we need to define $f$ near the $p_F$ only. So, we approximate $\vec{p} = p_F \hat{p}$ and $\vec{p}' = p_F' \hat{p}'$, where $\hat{p}$ and $\hat{p}'$ are the unit vectors on the Fermi surface.

→ Next we assume a spherical Fermi surface. This approximation does not hold well for large Fermi surface where the discrete crystal rotational symmetry becomes important, and we have to expand the Landau parameter in terms of the point/space group symmetry representation of the theory. Here we assume a continuous rotational symmetry, which is valid for electron gas, and also works fine for smaller Fermi surfaces.

In such a rotationally invariant system, the interaction term $f^{(S,A)}_{p,p'}$ only depends on the relative angle between the two momenta, say, $\theta$:

$$\vec{p} \cdot \vec{p}\,' = p_F^2 \cos\theta$$

And we have
$$f^{(S,A)}_{p,p'} \equiv f^{(S,A)}(\cos\theta) \quad \text{--- (11a)}$$

• A peridic function, which only depends on $\cos\theta \in (-1,1)$ be expanded on the basis of Legendre polynomials $P_\ell(\cos\theta)$ as

$$f^{(S,A)}(x) = \sum_{\ell=0}^{\infty} f_\ell^{(S,A)} P_\ell(x), \quad x = \cos\theta \quad \text{--- (12a)}$$

where the coefficients $f_\ell^{(S,A)}$ are called the Landau parameter, as given by

$$f_\ell^{(S,A)} = \frac{2\ell+1}{2} \int_0^1 dx\, f^{(S,A)}(x)\, P_\ell(x) \quad \text{--- (12b)}$$

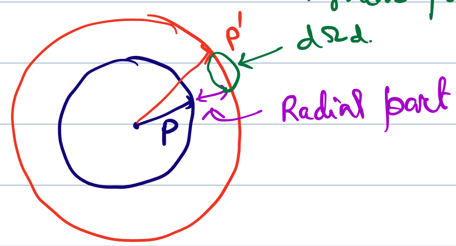
where $\ell\hbar$ = total angular momentum change during the interaction.

⊛ The self energy term in eq(7) is now defined as

$$\Sigma^{(S,A)}_{p,\sigma} = \sum_{p'\sigma'} f^{\sigma\sigma'}_{pp'}\,\delta n_{p'\sigma'}$$

$$= \frac{V}{(2\pi)^d}\sum_{\sigma'}\int_0^\infty p'^{d-1}dp' \int \underset{\uparrow \text{ solid angle in } d\text{-dim.}}{d\Omega'_d}\; \delta n_{p'\sigma'}\; f^{\sigma\sigma'}(\cos\theta). \quad \text{--(13 a)}$$

$$= \frac{V}{(2\pi)^d}\sum_{\sigma}\int_0^\infty \underbrace{p'^{d-1}dp'}_{\text{Along radial to FS}} \int d\Omega_d\; \delta n_{p'\sigma'} \underbrace{\underbrace{\sum_{\ell=0}^{\infty} f^{\sigma\sigma'}_{\ell}\int d\Omega'_d\, P_\ell(\cos\theta)}_{\text{From eq(12 a)}}}_{\text{Angular part of scattering.}} \quad \text{--(13 b)}$$




- Although it looks like we can simply do the two integrals and obtain a self energy, but note that $\theta$ is a relative angle between $p$ & $p'$, and $\delta n_{p'\sigma'}$ is actually the quasi-particle density after including the self-energy correction, i.e., the interaction effect. In fact, this equation is a self-consistent equation, which can be computed numerically.

- In this way of writing the above self-energy helps us distinguish between the excitations along the radial direction, i.e., across the equal energy contours, and across the angular direction, i.e., on the equal energy contour. Clearly, for those response functions which involves external energy costs, such as specific heat, charge/spin compressibility, the quasiparticles are excited only radially, and hence the angular part does not contribute. On the other hand, for various transport, which involves momentum scattering, the angular part contributes.

- For the case, when $\delta n_{p'\sigma'}$ is independent of the angle, ie, the quasiparticle excitations are only radial to the Fermi surface, we take $\delta n_{p'\sigma'}$ outside the angular integral. Then we can perform the angular integral exactly, we we get

$$\sum_{\ell=0} \int d\Omega_d \, f_\ell^{\sigma\sigma'} P_\ell(\cos\theta) = f_{\ell=0}^{\sigma\sigma'} \quad \text{[from eq (12a)]}$$

ie, the isotropic part of the interaction contributes only here

Then we have:

$$\boxed{\Sigma_{p,\sigma} = \frac{V}{(2\pi)^d} \sum_{\sigma'} f_0^{\sigma\sigma'} \int_0^\infty p'^{d-1} dp' \, \delta n_{p'\sigma'} .} \quad \text{(14a)}$$

- At $T=0$, all states upto the Fermi level is filled, ie, $\delta n_{p'\sigma'} = 1$ for $p \leq p_F$ and hence we get

$$\Sigma_{p,\sigma} = n \sum_{\sigma'} f_0^{\sigma\sigma'} \qquad n = \text{quasiparticle density}$$

$$\text{or,} \quad \boxed{\Sigma_p^{(S,A)} = n f_0^{(S,A)}}$$

This is a constant energy which is absorbed in the chemical potential.

- Now we can define the renormalization factor $Z$ in terms of $F_\ell^{(S,A)}$ easily. From eq (8), we have

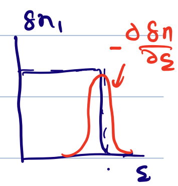
$$Z = 1 + \frac{\partial \Sigma_p^\sigma}{\partial \varepsilon_p}\bigg|_{p_F} = 1 + \sum_{p\sigma'} \left[ f_{pp'}^{\sigma\sigma'} \frac{\partial(\delta n_{p'\sigma'})}{\partial \varepsilon_{p\sigma}}\bigg|_{\varepsilon^0_{p\sigma}} + \frac{\partial f_{pp'}^{\sigma\sigma'}}{\partial \varepsilon_{p\sigma}}\bigg|_{\varepsilon^0_{p\sigma}} \delta n_{p'\sigma'} \right] \quad \text{--(15a)}$$

Clearly the first term comes from the change in density and the 2nd term is due to change in the potential. For most cases, the first term dominates, especially if the scattering is momentum conserved.

**Case I** Now, the energy fluctuation corresponds to going from one constant energy surface to another, i.e., the Fermi surface expands only radially. So, we have

$$Z = 1 + \frac{V}{(2\pi)^d} \sum_{\sigma'} f_0^{\sigma\sigma'} \int_0^\infty p'^{d-1} dp' \underbrace{\frac{\partial(\delta n_{p'\sigma'})}{\partial \varepsilon_{p\sigma}}\bigg|_{p_F}}$$

$$= 1 + \sum_{\sigma'} f_0^{\sigma\sigma'} \underbrace{\frac{V}{(2\pi)^d} \int_0^\infty p'^{d-1} dp' \, -\delta(\varepsilon_{p\sigma})}_{d(0)} \quad \text{at } T=0.$$




$$= 1 - d(0) \sum_{\sigma} f_0^{\sigma\sigma'}$$

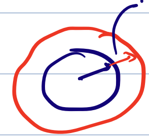
$$\therefore \; Z = 1 - F_0^{\sigma\sigma'}$$

where we define the Landau Parameter $\boxed{F_\ell^{\sigma\sigma'} = d(0) f_\ell^{\sigma\sigma'}}$

In terms of spin, we write

$$\boxed{Z^{(A,S)} = 1 - F_0^{(A,S)}} \quad \text{--(15b)}$$

$$|\delta p| = \frac{\partial(\delta n_p)}{\partial p} \delta p$$

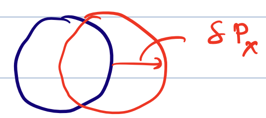


* When $F_0^{(A,S)} = 1$, one obtains an instability, called the Stoner Instability. Then one obtains a phase transition.

* $Z$ or $F$. roughly compares between the non-interacting (K.E) and interaction (P.E) term. For $0 < Z < 1$, K.E dominates, $Z=0$ says P.E. takes over.

Case II The above results of self-energy and renormalization factor arise where there is only density variance due to energy change for specific heat, compressibility etc. Here the first term in eq (15a) only contributed.

Now we want to consider the case where the system is boosted such that there is an overall momentum / current in the system. In this case, the FS is not expanded / contracted, but only shifted to the right. Therefore, the first term in eq (15a) is zero, only the 2nd term contributes.




The second term comes from the momentum being absorbed / gain by the change in the interaction potential $\partial f / \partial p$. This term is like a "force" in the momentum space, which is defined by the gradient of the potential $f$ in the momentum space. Therefore, we want to see here the change in momentum coming from the interaction term. Since near the Fermi level we have $\varepsilon_p \approx \hbar v_F p$, so express the quasiparticle energy term

$$\varepsilon_{p,\sigma} = \varepsilon^0_{p,\sigma} + \Sigma_{p,\sigma}$$

$$\Rightarrow \hbar v_F p \approx \hbar v_F^0 p + \left.\frac{\partial \Sigma_{p,\sigma}}{\partial p}\right|_{p_F} p + O(p^2).$$

$$\text{or,} \quad p = \underbrace{\frac{v_F^0}{v_F} p}_{p_0} + \underbrace{\frac{1}{\hbar v_F} \left.\frac{\partial \Sigma_{p,\sigma}}{\partial p}\right|_{p_F} p}_{-\frac{e}{c}\vec{A}} \quad \text{---(15d)}$$

[ Note that we have ignored $\Sigma_{p=p_F}$ term, because its absorbed to shift the chemical potential such that $p_F = p_F^0$. ]

Interestingly, the 2nd term can also be interpreted as an emergent vector potential in analogy with the gauge transformation $\vec{P} \to \vec{P}_0 - \frac{e}{c}\vec{A}$. The vector potential can then be converted into electric / magnetic field which balances the electric / magnetic dipole created by the shift of the Fermi surface. This can also be understood from the Galilian invariance of the system, which Landau himself used for the derivation. In that case, we go to the moving reference of frame of the $p'$- state (particle) and look at the $p$-state (particle) of our interest. Then we see that the $p$-state's momentum has changed in this frame to be $p \to p_0 + p''$, where $p''$ is the center of mass momentum.

Now, in this computation of $\frac{\partial \Sigma}{\partial p}$, we set $\frac{\partial(\delta n)}{\partial p} = 0$, but only consider $\partial f/\partial p$ term in eq(15b). (Note that in some books, it is considered how many quasiparticles are shifted, i.e, $\partial(\delta n)/\partial p$, and ignore $\partial f/\partial p$. Both procedure are of course equivalent). So, we have,

$$\frac{\partial \Sigma_{p,\sigma}}{\partial p}\Bigg|_{p_F} = \frac{V}{(2\pi)^d} \sum_{\sigma'} \underbrace{\int p'^{d-1} dp' \, \delta n_{p'\sigma'}}_{n} \sum_{l=0}^{\infty} f_l^{\sigma\sigma'} \int d\Omega_d \, \underbrace{\frac{\partial P_l(\cos\theta)}{\partial p}\Bigg|_{p_F}}_{\frac{1}{p_F}\delta_{l,1}}$$

$$= \frac{n}{p_F} \sum_{\sigma'} f_1^{\sigma\sigma'} = \frac{n}{p_F} f_0^{(s,A)}. \qquad \text{--- (15e)}$$

[ $\frac{\partial P_l(\cos\theta)}{\partial p} = \frac{\partial P_l(\cos\theta)}{p_F\, \partial(\cos\theta)}$ $= 0$ for $l = 0$

$= 1$ for $l = 1$.

$= P_l'(\cos\theta)$

Then $\int d\Omega\, P_l'(\cos\theta) = 0$ for $l \neq 1$ due to orthogonality of the Legendre polynomials. ]

Now, renormalized density of state in $d(0) = \frac{m^* p_F}{3\pi^2}$

$n = \frac{p_F^2}{3\pi^2}$ (in 3D). So, $n / p_F = (d(0)/m^*)\, p_F = v_F\, d(0)$

Then we have $\left.\frac{\partial \Sigma}{\partial p}\right|_{p_F} = v_F \underbrace{d(0) f_1^{(S,A)}}_{F_1^{(S,A)} \text{ (define)}} = v_F\, F_1^{(S,A)}$

Next going back to eq (15d), we get at $p = p_F$.

$$\cancel{p_F} = \frac{v_F^0}{v_F} \cancel{p_F} + \frac{\cancel{v_F}}{\cancel{v_F}} F_1^{(S,A)} \cancel{p_F}.$$

Therefore, $\frac{v_F^0}{v_F} = 1 - F_1^{(S,A)}$

$$\text{or, } \boxed{v_F / v_F^0 \approx 1 + F_1^{(S,A)}} \quad \text{--- (15e)}$$

Since momentum $p_F = v_F^0 m^0 = v_F m^*$, so,

$$\boxed{\frac{m^*}{m} = \frac{v_F^0}{v_F} = 1 - F_1^{(S,A)}} \quad \text{--- (15f)}$$

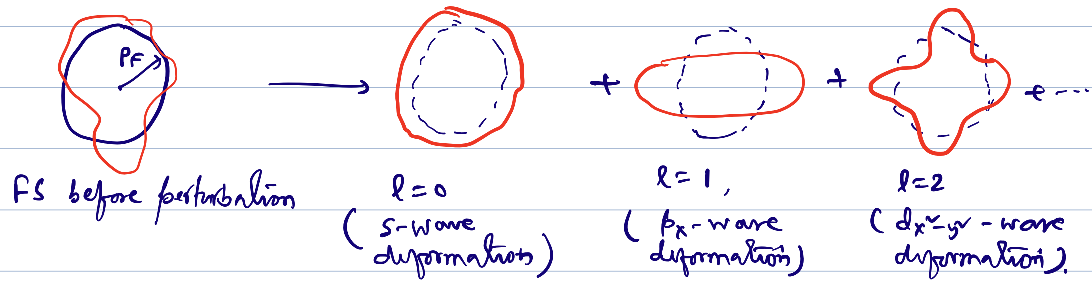
Therefore, the mass is enhanced or the Fermi velocity is reduced by the dipolar back reaction of the interaction

At $F_1^{(S,A)} = -1$, i.e., at $d(0) f_1^{(S,A)} = -1$ the velocity $v_F \to 0$ or mass $m^* \to \infty$. This also corresponds to a phase transition point, called the Mott insulating transition. Needless to say, the Fermi-liquid theory breaks down here. This is actually the same instability that obtained in terms of $f_0^{(S,A)}$, called the Pomeranchuk instability - which results in a permanent distortion of the Fermi surface - Called Pomeranchuk distortion.

→ For $l=2$ case, similar instability arises, which is called nematic phase.

## 5.4: Calculation of Response Functions with Landau parameters.

We will only the focussing on change in density, energy etc due to change in Fermi surface via perturbations. Since number of electron is conserved, and hence Fermi surface volume is conserved, so, the Fermi surface can only be deformed (except for applied electrostatic potential which causes $\delta\mu$ and hence $\delta n$). The deformation of the Fermi surface is decomposed in angular momentum components, as pictorically demonstrated below:




There are two terms in the Hamiltonian which participate in the deformation – the kinetic energy term ($\varepsilon_p^0$) which is a single particle term and is trivial and the interaction term – which depends on the density of other particles and the Landau parameter. We have decomposed the Landau parameter in the angular momentum basis in eq (16), so that it will be easier to discern each angular momentum contribution. Lower angular momentuntum means large-angle (like large wavelength) fluctuation and is lower in energy, because locally its slow fluctuation. This way we will only keep the lowest angular momentum (i.e. the leading) term in $F_l^{(S,A)}$ for a given response function.

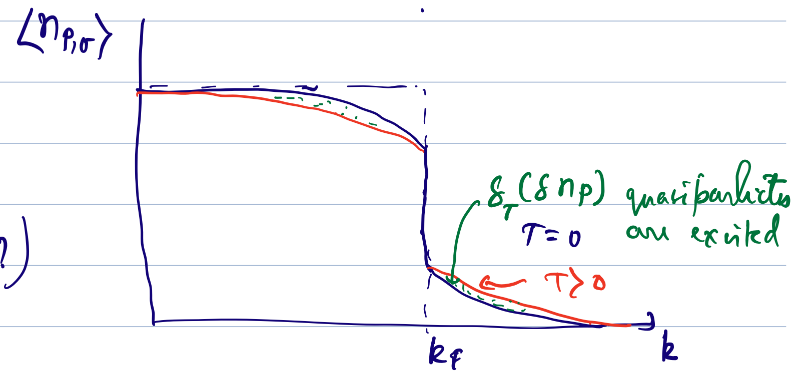
## ⊛ Specific heat :

There are 4,5 typical response function that we often measure and compute

→ $C_V$ - specific heat is a response function to temperature change and measures the energy fluctuation. The linear response theory of specific heat is

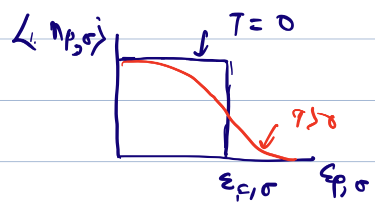
$$\delta E = C_V \, \delta T .$$

During the thermal fluctuation, the quasiparticles are excited across the fermi-level which is captured by the Fermi-Dirac distribution function. For the quasiparticles case, the change in momentum density or momentum distribution function




(Is this correct?) ???

⊛ H.W.: Show that the quasiparticle energy distribution function in terms of quasiparticle energy $\varepsilon_{p,\sigma}$ remains the same as $f(\varepsilon_{p,\sigma}) = \left(e^{(\varepsilon_{p,\sigma}-\mu)} + 1\right)^{-1}$.



⊛ Since the energy distribution function $n_{p\sigma} = f(\varepsilon_{p,\sigma})$ remain the same, so, the entropy also remain the same. The entropy term takes the same form as to arise from the classical probability of occupying a state $(p,\sigma)$ being $n_{p,\sigma} = f(\varepsilon_{p,\sigma})$ and that of having it empty is $(1 - n_{p,\sigma})$. So, we get

$$S = -k_B \sum_{i \in \text{microstates}} p_i \ln p_i$$

$$= -k_B \sum_{p,\sigma} \left[ n_{p,\sigma} \ln n_{p,\sigma} + (1 - n_{p,\sigma}) \ln(1 - n_{p,\sigma}) \right]$$

Then we can compute specific heat $C_V = \frac{1}{T} \left( \partial S / \partial T \right)_V = \frac{\partial \varepsilon}{\partial T}\Big|_V$ and we obtain

$$C_V = \gamma T, \quad \text{where } \gamma = \frac{\pi^2}{3} k_B^2 \, d(0)$$

$$= \text{Sommerfeld coefficient.}$$

$$= \frac{m^* p_F}{3\hbar^3} k_B^2 . \qquad \text{--- (15)}$$

[ H.W. Derive eq 15 ].

→ $\chi_c$ – Charge susceptibility or compressibility. This is a response function to electrostatic potential, which causes density fluctuations. We are here looking at $\chi_c(q\to 0, \omega\to 0)$, ie, the global (spatially uniform) electrostatic potential - which is the change in chemical potential $\delta\mu$, and measuring the change in number of electrons $\delta n_c$, which are related to each other by the linear response theory as

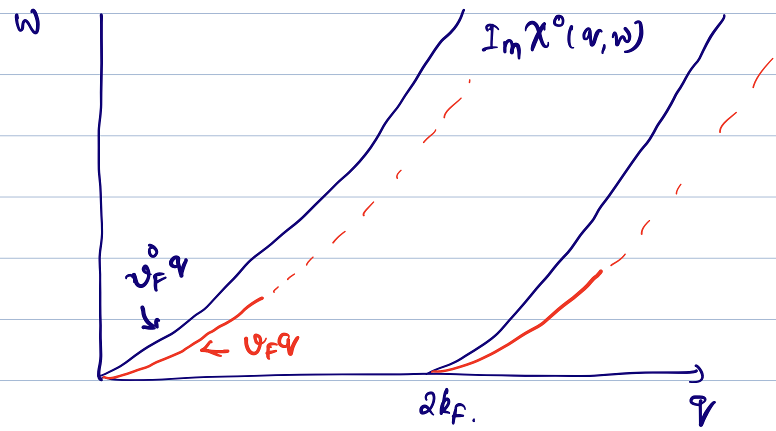
$$\delta n_c = \chi_c \, \phi_{ext} \Rightarrow \delta n_c = -\chi_c \, \delta\mu \Rightarrow \boxed{\chi_c = -\frac{\partial n_c}{\partial \mu}}$$

• The effect of the Landau interaction is to change the electronic structure by $\underline{\varepsilon_{p,\sigma} = \varepsilon^0_{p,\sigma} + \Sigma_{p,\sigma}}$, and as we established that the energy distribution function retains the same form, so, the response function/charge susceptibility just modified to be

$$\chi(q,\omega) = \sum_{p,\sigma} \frac{f(\varepsilon_{p\sigma}) - f(\varepsilon_{p+q,\sigma})}{\omega + \varepsilon_{p,\sigma} - \varepsilon_{p+q,\sigma}} \quad \text{--- (16)}$$

• We notice an important fact here that the effect of the Landau interaction is not to give a RPA like correction to the non-interacting / bare susceptibility and to produce a Plasmon mode (collective mode), but rather to replace the electrons by a collective charge puddle which behave like free-particles (quasi particles).

Therefore, in the response function, we will only have the same particle hole continuum, but only its slope will be modified.




→ Only the slope of the particle-hole continuum will be renormalized to $v_F = Z v_F^0$.

This zero-energy particle hole continuum of Landau's interacting quasiparticle is called the zero-sound mode.

→ Of couse, to keep this in mind that the Fermi liquid approximation is only valid near the Fermi level, while at higher energy, it breaks down. Then one can obtain a RPA like expression for the quasiparticles and plasmon modes appear and also seen experimentally.

→ For the compressibility calculation, which is the result at $\omega \to 0$, $q \to 0$, recall that the ordering in limit is important. We first take the $\omega \to 0$, i.e. long time time before we make the electrostatic potential to be uniform ($q \to 0$) limit such that we can capture the density fluctuations near the Fermi level.

Because the expression of $\chi$ remain the same, hence the result is also the same, i.e.

$$
\begin{aligned}
\chi(q \to 0, \omega \to 0) &= \sum_p \frac{\partial f}{\partial \varepsilon_p} \frac{\cancel{(\varepsilon_{p+q} - \varepsilon_p)}}{\cancel{(\varepsilon_{p+q} - \varepsilon_p)}} \\
&= \sum_p \frac{\partial}{\partial \varepsilon_p} f(\varepsilon_p - \mu) = -\sum_p \frac{\partial f(\varepsilon_p - \mu)}{\partial \mu} \\
&= -\frac{\partial}{\partial \mu} \sum_p f(\varepsilon_p - \mu) = -\frac{\partial n}{\partial \mu} \\
&= d(0) = \text{density of quasiparticles} \\
&= Z^{-1} d_0(0).
\end{aligned}
$$

* What happens when $Z \to 0$, i.e., $1 + F_0^{(A,S)} \to 0$ or, $F_0^{(A,S)} \to -1$. This is actually a singular point, where the analytic continuity between $\varepsilon_{p,\sigma}$ & $\varepsilon^0_{p\sigma'}$ breaks down, and the Fermi liquid theory breaks down. In fact, this is also the phase transition point to a Pomeranchuk phase for charge core ($F_0^{(S)}$) or to a Ferromagnetic phase for $F_0^{(A)}$. This singularity is called the Pomeranchuk instability and the Stoner instability, respectively.

## 5.5 Quasiparticle Lifetime:

One of the triumph of the Fermi-liquid theory is the long-lifetime of the quasiparticle in the low energy spectrum, i.e, near the Fermi level. This is because of the Pauli exclusion principles which put several restrictions on the available states near the Fermi level, irrespective of the interaction and its strength. Our claim was that the lifetime $\tau \sim 1/\varepsilon^2$, $\varepsilon =$ quasiparticle energy. We can now justify this claim.

The self energy $\Sigma$ is actually complex, which can be seen if we include time-dependent response function, viz, as the so-called Green's function approach. The real part of the self-energy is calculated here in the Fermi liquid case, and it gives the correction to the electronic dispersion due to interaction. The imaginary part gives an imaginary contribution to the energy, which means, it gives the inverse lifetime to the quasiparticle in a given state.

Because the self-energy is analytic, the real and imaginary parts are related to each other by the Kramers Kronig relation. The KKR however involves integration over the entire energy range, and the present Fermi-liquid method is valid only near the low-energy region. Therefore, we cannot directly use this theorem to obtain the imaginary part here. But using the symmetry argument, we can predict its behavior near the Fermi level.

$\Sigma_k$ is linear in $\varepsilon_k$ near the Fermi level, and its slope is given by $Z$ as $\Sigma_k \sim (Z-1)\,\varepsilon_k$. Therefore, its and odd function in $\varepsilon_k$. In the KKR theory, we saw that an odd real part gives an even imaginary part. The lowest order even term is $\varepsilon_k^2$ and hence we must have

$$\Sigma_k'' = \frac{\hbar}{\tau_k} \sim \varepsilon_k^2 .$$

H.W. Using Fermi Golden rule, show that the Pauli-blocking principle indeed gives a $\tau_k^{-1} \sim \varepsilon^2$ term in the Fermi liquid theory.