Chapter 8: Transport (Mainly Boltzmann transport equation)

1. Boltzmann transport equation
2. Relaxation time approximation
3. Conductivity
4. Impurity scattering
5. Electron phonon scattering
6. Electron - electron scattering

Refs: 1. Lecture notes by Brown. (Sec 22)
2. J. Solom (Vol 2) chapter 24
3. P. Phillips chapter 11.5.

Generally, every thermodynamically extensive quantity or order parameter that we study, such as number density, charge density, magnetization, energy density, heat density, entropy density, etc, has an associated current density, which is defined by the density times velocity. (Of course, some of the densities are related to each other, such as charge density and current density are related to each other by charge 'e' and entropy density is related to heat density by the proportionality constant T etc). Generally, what does that mean is that time derivative of the density of the quantity in hand is defined by the spatial gradient of the corresponsiding current density, is, the continuity equation. The continuity equation is a manifestation of the conservation of energy and momentum. But as the electron undergoes scattering with external perturbation (such as impurity, phonon) they loose/gain momentum and/or energy and hence the continuity equation is modified to the Boltzmann equation that we want to derive here.

In condensed matter we measure current density and the corresponding response functions (such as electrical conductivity, thermal conductivity) to the corresponding external perturbations in a linear response theory. The conductivity of present interest is related to the current-current (retarded) commutator which we have computed already in a fermionic system and arise from the energy absorption due to particle-hole continuum. But this calculation does not include the impurity scattering and electron-phonon scattering which does not conserve the energy and momentum of electrons.

A full quantum mechanical formulation of transport in which we compute the evolution of an electronic state with a Hamiltonian that incorporates energy, momentum non-conservation is difficult and requires non-Hermitian Hamiltonian formalism. The Drude model is a fully classical one in which one studies the evolution of the electron's coordinate $(\vec{r})$ under a "dissipative force" which models scattering to impurity, nucleus etc. A dissipative force, by definition, is proportional to the velocity of the electron as defined by $\vec{F} \propto -\vec{v} = -\eta\vec{v}$, where η is like the friction, arising due to collision of electrons with impurity, nucleus etc. Here it is customery to express η in terms of "relaxation time" τ (or mean free path $l = v\tau$) between the two collisions (on average). This can be defined using dimensional analysis (since Force $= m\dot{v}$) to be $\eta = m/\tau$, where m is the electron's mass. For an applied electric field, the electric force $\vec{F} = -e\vec{E}$ equates to the dissipative force, in a steady state, which gives

$$-e\vec{E} = -\frac{m}{\tau}\vec{v} = -\frac{m}{ne\tau}\vec{J}. \quad [\because \vec{J} = ne\vec{v}].$$

Then using Coulomb's law $\vec{J} = \sigma\vec{E}$, we get

$$\sigma = \frac{ne^2\tau}{m} \qquad -(1).$$

τ or mean free path is the only parameter in this theory. In the Fermi liquid theory and in the electron-phonon coupling chapter, we argued that if the relaxation time is governed by the electron's life time in a momentum (Bloch) state. This is captured by the imaginary part of the self-energy. Alternatively, τ can be calculated by using the Fermi Golden rule, evaluating the matrix element of scattering of a state to another state under a given potential. At finite temperature, the thermal energy k_BT also contributes to the relaxation time as $\tau = \hbar/k_BT$.

The Drude model, despite being a fully classical model, is quite successful in the case when the mean-free path $l = v_F\tau$ is much larger than their de-Broglie wavelength such that quantum effect is negligible. To improve the Drude model, we have to develop a quantum theory of the evolution of the wavefunction $\psi(\vec{r})$ under a dissipative force. This is very hard and is under active current research. Here we will consider an intermediate route - the semiclassical route - in with we will study the evolution of the occupation density $n(\vec{r})$ of electrons under a dissipative force. Occupation density itself is a quantum operator and can be written in terms of the field operator $\psi(r)$ as $n(r) = \psi^\dagger(r)\psi(r)$. We are however going to study its evolution as a "classical" probability for occupying a single particle state, but we will enforce the quantum statistics by restricting to single occupancy for electrons. Hence its a semi-classical treatment.

If we are in some single particle eigenstate and that the system is in some thermal equilibrium, then the occupation density is simply the Fermi-Dirac distribution function $n = f(k) = [\exp(\epsilon_k - \mu)/k_BT + 1]^{-1}$. But as we apply an electrostatic potential difference $\vec{E} = -\vec{\nabla}\phi$, it gives a spatial variation of the chemical potential $\mu(r)$ or as a temperature gradient is applied, we have $T(r)$. To the first approximation, we incorporate the r-difference in μ & T in the same occupation density as

$$n(r,k) = f(r,k) = \left[\exp\left(\varepsilon_k - \mu(r)\right)/k_B T(r) + 1\right]^{-1} \quad \text{----(2)}.$$

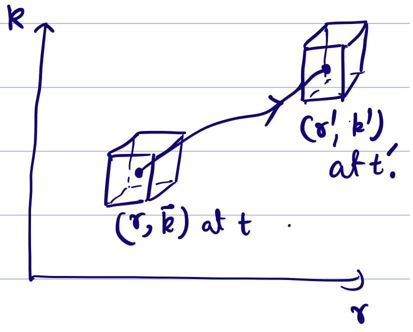
This is actually a crude approximation in which we assume the energy eigenvalue ε_n remains the same. We will see below that this approximation is valid for slow variation of μ & T in space. In fact, the occupation density can also be time-dependent, in general, for non-equilibrium case. Then such a distribution functions $f(r,k,t)$ is called the non-equilibrium distribution function. It looks strange to define the distribution function to be a function of both position and momentum. Because, in cond matter, we often define the states in the momentum space only by the usage of Bloch states. This definition is justified when we use a wave packet form of the wavefunction, not just the Bloch state.

Therefore, $f(r,k,t)$ is like a probability distribution defined in the Phase space (r,k). The time evolution of a number density in the phase space was derived by Boltzmann in 1873 or so when the quantum mechanics was not formulated. This is generally called the Boltzmann transport equation. We are going derive this Boltzmann transport equation for the fermion distribution function here in the phase space for electrostatic potential as well as with a temperature gradient, and thereby compute electrical and thermal conductivity due to some scattering mechanism which causes dissipation.

Boltzmann Transport Equation :

In the phase space, $f(\vec{r},\vec{k},t)$ is like the ensemble density, defining the number of electrons present at time t in a phase space volume element $d^3r\, d^3k$ around a point $(\vec{r},\vec{k})$.

In the Boltzmann transport formalism, the total number of particle (N) is conserved, which is given by



$$2\int f(r,k,t)\,\frac{d^3r\,d^3k}{V\,\underline{(2\pi)^3}_{\;V}} = N.$$

V is the real space volume and $(2\pi)^3/V$ is the momentum space volume. The factor '2' is introduced for spin.

We assume that the measure of the integral $d^3r\, d^3k$ remain invariant in time. This is the Liouville theorem. Liouville's theorem is generally applicable in "flat" phase space, and is only violated if the phase space is "curved" which happens if we apply a magnetic field, or if the phase space is topologically non-trivial or in general theory of relativity where the measure is defined with a space-time dependent metric. There the measure also evolves in time. Here we will assume "flat" phase space.

Thus, instead of studying how the particle's coordinates (r, k) evolves in time, we can simply focus "locally" at (r, k), and ask how the density $f(r,k,t)$ evolves in time. The total time-derivative of f is thus obtained to be

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\vec{r}}\cdot\frac{\partial f}{\partial \vec{r}} + \dot{\vec{k}}\cdot\frac{\partial f}{\partial \vec{k}} = \text{Dissipation/ Diffusion term.} \qquad --(1)$$

This is the Boltzmann equation.

In equilibrium, there is no net loss of occupation density and hence the Dissipation / Diffusion like term on the R.H.S is zero. This gives the continuity equation. But when collisions are included, the occupation density changes within the volume element and the change in the occupation density is denoted by $\left.\frac{\partial f}{\partial t}\right|_{\text{collision}}$. Clearly, $\left.\frac{\partial f}{\partial t}\right|_{\text{coll}}$ results from the net inward flow from the outward flow of electrons in the volume element in the time interval dt

Returning back to eq (3), $\dot{\vec{r}}$ and $\dot{\vec{k}}$ are obtained from the Hamiltonian equation of motion $\dot{\vec{r}} = \partial H/\partial \vec{p}$ and $\dot{\vec{k}} = -\partial H/\partial \vec{r}$ $(\hbar = 1)$.

And substituting them we have

$$\frac{\partial f}{\partial t} + \frac{\partial H}{\partial \vec{p}} \cdot \frac{\partial f}{\partial \vec{r}} - \frac{\partial H}{\partial \vec{r}} \cdot \frac{\partial f}{\partial \vec{p}} = \left.\frac{\partial f}{\partial t}\right|_{\text{coll}}$$

$$\Rightarrow \frac{\partial f}{\partial t} = \underbrace{\{H, f\}_{\text{P.B.}}}_{\text{External force / perturbation goes there}} + \underbrace{\left.\frac{\partial f}{\partial t}\right|_{\text{coll}}}_{\text{Interation term that relaxes the system to equilibrium}} \quad \text{--- ⑤}$$

[This time evolution of a function in phase space $f(r, k, t)$ governed by the Poisson bracket in classical mechanics is replaced with the commutator for operator in quantum mechanics. In quantum mechanics we work with density matrix $\rho = |\psi\rangle\langle\psi|$ instead of the occupation density.]

There is a longer derivation of the $\dot{\vec{r}}$ and $\dot{\vec{k}}$ term, which can be found in the book of Girvin and Yang. We only use the semiclassical result given by:

$$\dot{\vec{r}} = v_k = \frac{1}{\hbar} \frac{\partial \varepsilon_k}{\partial k} \quad \text{--- (6a)}$$

[Group velocity of a wave packet].

and $$\hbar\dot{\vec{k}} = \text{Lorentz Force } F = -e\left[E(\vec{r},t) + \frac{1}{c}\vec{v}_k \times \vec{B}(r,t)\right] \quad \text{-- (6b)}$$

(we are actually not going to work with magnetic field term which gives Hall effect. However we will include it for the time being).

Substituting eq (6a) and (6b) in eq(5) we have

$$\boxed{\frac{\partial f}{\partial t} = -\vec{v}_k\cdot\underbrace{\vec{\nabla}_r f}_{\text{Density variation in real space}} - \frac{1}{\hbar}\vec{F}\cdot\underbrace{\vec{\nabla}_k f}_{\text{Density variation in momentum space.}} + \frac{\partial f}{\partial t}\Big|_{coll.} \quad \text{—(7a)}}$$

This is the semi classical Boltzmann transport formula for electrons in a metal.

- For only <u>electrical conductivity</u> (σ), the density is uniform in real space and hence $\nabla_r f = 0$. There is also no explicit time-dependence in f and hence $\partial f/\partial t = 0$. So, we are left with
$$\vec{F}\cdot\vec{\nabla}_k f = -\hbar\frac{\partial f}{\partial t}\Big|_{coll.}$$

- For thermal conductivity (κ), f has a spatial variation due to spatial variation of temperature $T(r)$ and we have
$$\vec{\nabla}_r f = \frac{\partial f}{\partial T}\vec{\nabla}T .$$
Then we have
$$\hbar\,\vec{v}_k\cdot\vec{\nabla}T\left(\frac{\partial f}{\partial T}\right) = \frac{\partial f}{\partial t}\Big|_{coll}$$

* The linearized Boltzmann Equation

In the next step, we assume that on average $f(r,k,t)$ is a slowly varying function in the phase space. Before, the external potential is applied, the system is in thermal equilibrium as $f(r,k,t) = f_0(k)$ = Fermi-Dirac distribution function. (The thermal equilibrium is achieved via including all the collision/scattering process present in the system). With the external potential (electrostatic potential and/or temperature gradient), the occupation density deviates slowly w.r.to its equilibrium value. So, we define:

$$f(r,k,t) = f_0(\underset{\uparrow}{r},k) + \delta f(r,k,t) \quad \text{---}(8)$$

[A r-dep can be kept for the thermal case $T(r)$]

- By the definition of equilibrium, the system reached to the thermal equilibrium due to collision/scattering, and hence

$$\left.\frac{\partial f_0}{\partial t}\right|_{coll} = 0$$

- Then from eq (7a), ignoring the explicit time dependence, we get

$$\boxed{\left.\frac{\partial(\delta f)}{\partial t}\right|_{coll} = \vec{v}_k \cdot \underbrace{\frac{\partial f_0(r)}{\partial \vec{r}}} + \frac{1}{\hbar}\vec{F} \cdot \frac{\partial f_0(k)}{\partial \vec{k}}} \quad -(9)$$

Generally arises from $T(r)$ in f for thermal conductivity calculation

Generally, the force due to (uniform) electrostatic potential

$$= \vec{v}_k\cdot\Big(\underbrace{\nabla_r T\,\frac{\partial f_0}{\partial T}}_{\text{since } \frac{\partial f_0}{\partial r} = -\frac{\partial T}{\partial r}\frac{\partial f_0}{\partial T}}\Big) + \vec{F}\cdot\Big(\underbrace{v_k\cdot\frac{\partial f_0}{\partial \varepsilon_k}}_{\text{since } \frac{\partial f_0}{\partial k} = \frac{\partial f_0}{\partial \varepsilon_k}\frac{\partial \varepsilon_k}{\partial k} = \hbar v_k\frac{\partial f_0}{\partial \varepsilon_k}}\Big)$$

$$= \vec{v}_k\cdot\Big[-\vec{\nabla}_r T\,\underbrace{\frac{\partial f_0}{\partial T}}_{\frac{\varepsilon-\mu}{T}\frac{\partial f_0}{\partial \varepsilon}} + \underset{-e\vec{E}}{\underset{\uparrow}{\vec{F}}}\,\frac{\partial f_0}{\partial \varepsilon}\Big].$$

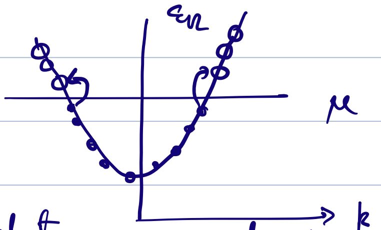
$$= -\vec{v}_k\cdot\Big[\frac{\varepsilon-\mu}{T}\vec{\nabla}T + e\vec{E}\Big]\frac{\partial f_0}{\partial \varepsilon} \quad -(9b).$$

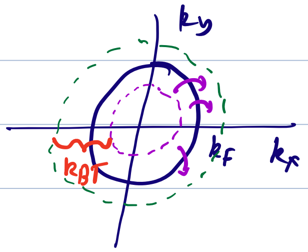
As $T\to 0$, $\partial f_0/\partial\varepsilon \to -\delta(\varepsilon-\mu)$. So, we have

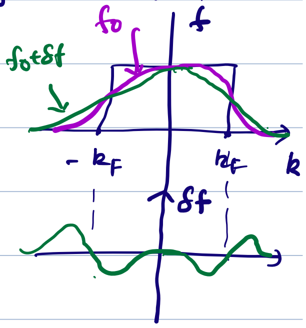
$$\boxed{\left.\frac{\partial(\delta f)}{\partial t}\right|_{coll} = v_k\cdot\Big[\frac{\varepsilon-\mu}{T}\vec{\nabla}T + e\vec{E}\Big]\delta(\varepsilon-\mu) \quad -(9c).}$$

- Let us look into how the Fermi surface response to the two perturbation on Q.A.S (also discussed in the Fermi liquid chapter).

→ Temperature gradient: It excites electrons from the Fermi sea to outside the Fermi sea.

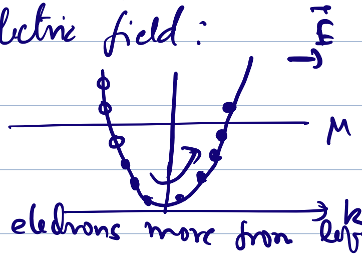


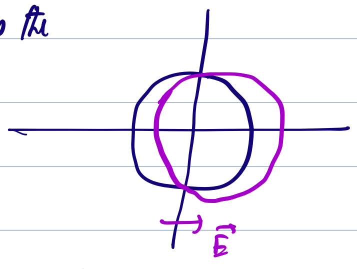


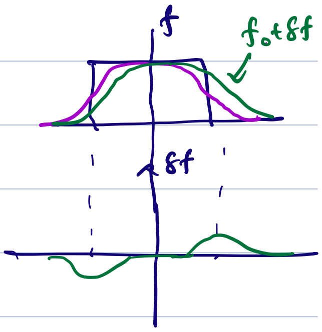


electrons move out of the FS.

→ On the other hand, electric field boost the electrons and hence the Fermi surface shifts towards the electric field:







electrons move from left to right

⊛ The relaxation time Approximation

Finally we introduce another important approximation, called the Relaxation-time Approximation. Here we assume that the distribution function relaxes to equilibrium one f_0 after a charateristic time. τ. This means
$$\left.\frac{\partial f}{\partial t}\right|_{coll} \rightarrow -\frac{\delta f}{\tau} \qquad --(10a)$$

(More generally, the relaxation time should be position and momentum dependent $\tau(r,k)$ which we will consider later).

Clearly, how long does the system takes to relax to equilibrium depends on the scatting mechanism / collision process that the system has (ie, impurity, electron-phonon coupling, electron-electron interaction). For an integrable model, where $f_0(k) = n(k)$ are the eigenstates of the Hamiltonian, ie, $\{H, f\}_{P.B} = 0$ for the classical case, there is no scattering mechanism and hence the system does not relax at all. Here $\tau \rightarrow \infty$.

So, our job below is to compute τ due to impurity, electron-phonon and due to electron-electron interaction. For the two later cases, it is tempting to assume the relaxation time τ is the same as the electron's lifetime that we derived from the imaginary part of the self-energy or from the Fermi Golden rule. They are often the same, but there can be some difference.

By substituting, eq (10a) in eq (9b), we obtain the change in the distribution function due to the external perturbation, in terms of the relaxation time τ as:

$$\boxed{f(k,r) = f_0(k,r) - \tau \vec{v}_k \cdot \left[\frac{\varepsilon_k - \mu}{T} \vec{\nabla} T + e \vec{E} \right] \quad -(10b)}$$

(This is analog to the linear response theory in that the induced occupation density fluctuation is linearly proportional to the external perturbation.)

Once we know τ, we can evalue the induced density and obtain the conductivity. τ is a parameterization of all the interaction, scattering with impurity, electron-phonon coupling term in a single parameter. In fact, in terms of this parameter τ, eq (10b) recovers the Drude formalism. In the reminder of this chapter, we want to discuss formalism/approximation to compute τ.

⊛ Low-temperature (T→0) conductivity : Recovery of Drude model.

By keeping the relaxation time τ as a parameter and going to $T\to 0$ limit in eq (9c), we can recover the Drude model's result for the electric and thermal conductivities σ, κ, respectively and the Wiedemann-Franz law.

→ The electrical current is defined as

$$
\begin{aligned}
\vec{J} &= -e\sum_k \langle \vec{v}_k\rangle f_k \\
&= -e\sum_k \langle \vec{v}_k\rangle (f_0+\delta f) \\
&= -e\sum_k \langle \vec{v}_k\rangle \delta f_k \qquad \text{As } \langle v_k\rangle = 0 \text{ in equilibrium} \\
&\qquad\qquad --(11a)
\end{aligned}
$$

For electrical conductivity, we set $\nabla T = 0$ in eq (9c). Then substituting eq (10), we get from eq (9c)

$$\delta f_k = \tau\,\vec{v}_k\cdot(e\vec{E})\,\delta(\varepsilon_k-\mu)$$

substituting this in eq (11a) we get

$$\vec{J} = e^2\tau\sum_k \underset{\uparrow}{\vec{v}_k}\,(\vec{v}_k\cdot\vec{E})\,\delta(\varepsilon_k-\mu).$$

← This v_k is along the field.

This v_k is along the measured current

Because of the δ-function, the states near the fermi level contributes. So, we replace $\langle v_k\rangle$ with an isotropic fermi velocity v_F. We also focus on longitudinal conductivity i.e., $J=\sigma\vec{E}$, so, we have

$$J_x = e^2\tau E_x v_F^2 \underbrace{\sum_k \delta(\varepsilon_k-\mu)}_{d(0)\ =\ \text{density of states}}$$

or, $$J_x = e^2 \tau E_x \underbrace{v_F^2 d(0)}$$

$$\frac{\varepsilon_F d(0)}{m} = \frac{n}{m} \quad \text{as} \quad d(0) = \frac{n}{\varepsilon_F}$$

(For 3D, multiply by 3 in $d(0)$ and for σ do devide by 3 for average $\sigma = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$.

(The factor of 2 for spin also drops out)

$$= \frac{n e^2}{m} \tau \, E_x \qquad -(11b)$$

Therefore, the electrical conductivity is $\sigma_{xx} = \frac{n e^2}{m} \tau$, which is the Drude conductivity.

→ Similarly, the thermal current is defined as

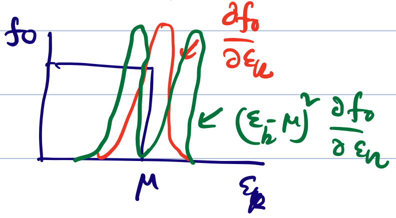
$$\bar{J}_e = \sum_k (\varepsilon_k - \mu) \langle v_k \rangle \delta f_k .$$

$$= -\frac{\tau}{T} \sum_k \vec{v}_k \, (\bar{v}_k \cdot \bar{\nabla} T) \, (\varepsilon_k - \mu)^2 \frac{\partial f^0}{\partial \varepsilon_k}$$

$$= -K . \bar{\nabla} T \qquad -(11c)$$

where K is the thermal conductivity tensor.

The function $(\varepsilon_k - \mu)^2 \frac{\partial f_0}{\partial \varepsilon_k}$ has a double peakek structure about μ :



To do this integral we can use the Sommerfeld expansion and the final result is

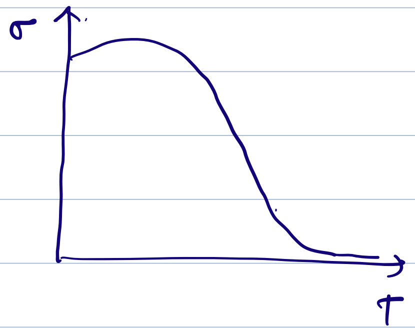
$$\boxed{K_{xx} = \frac{n\tau}{m}\left(\frac{\pi^2}{3}k_B^2\right)T} \qquad -(11d)$$

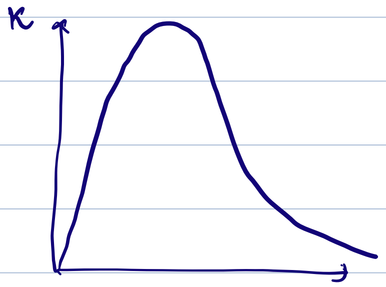
→ Then the Wiedemann-Franz law is maintained as

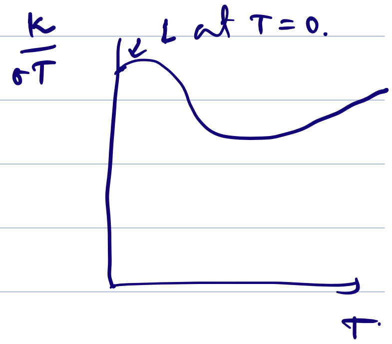
$$\boxed{\frac{\kappa}{\sigma T} = \frac{\pi^2 k_B^2}{3e^2}} = \text{Lorentz ratio } (L)$$

$$— (11e)$$

→ In a metal







⊛ Calculation of τ :

From the above discussion, an intriguing physical interpretation of the transport phenomena emerges – which is analogous to the Fluctuation-Dissipation Theory. With external perturbation ($\vec{E}$ or $\vec{\nabla} T$) we have excited/induced "δf" density at a given k state from its thermal equilibrium, and then asking, with collisions, it relaxes back to its equilibrium value ($\delta f = 0$) after a time scale 'τ'.

So, 'τ' measures how long an electron spends in the state k, σ, 'τ' is the lifetime of the electron in the k-state. τ is finite because the electron scatters from this state k to some other state 'k'' due to interaction / electron-phonon / impurity scattering. We have computed the life-time of electrons for the former two process as the imaginary part of the self energy Σ'' as

$$\tau_k \sim -\frac{\hbar}{\Sigma''_k} .$$

We can also evaluate τ using the Fermi-Golden rule. Depending on the scattering potential, there will be difference between the relaxation time and the lifetime of electrons in a system, but we will not look into those cases here. The relaxation time τ is often called the transport time, in which an external perturbation is applied, compared to electron's intrinsic lifetime.

(*) It is often useful to express $\left.\frac{\partial f}{\partial t}\right|_{coll}$ in terms of scattering amplitudes for some arbitrary potential. There are two scattering process: one which scatter an electron from the wave packet $\psi_k(r)$ to some other wave packet $\psi_{k'}(r')$ - resulting in a decrese in the occupancy at the (k,r) point in the phase space. This is the loss term. For this scattering to occur the initial state $\psi_k(r)$ must be filled while $\psi_{k'}(r')$ must be empty. If $W_{kk'}$ is the scattering probability then we have

$$\text{Loss} = \left.\frac{\partial f}{\partial t}\right|_{loss}(r,k) = \int \frac{d^3k'\,d^3r'}{(2\pi)^3} W_{kk'}\, f(r,k,t)\left(1-f(r',k',t)\right) \quad \text{--(12a)}$$

Similarly, the probability density increases (gain) for a reverse process

$$\text{Gain} = \left.\frac{\partial f}{\partial t}\right|_{gain}(r,k) = \int \frac{d^3k'\,d^3r'}{(2\pi)^3} W_{k'k}\left(1-f(r,k,t)\right) f(r',k',t) \quad \text{--(12b)}$$

The scattering is assumed to be instantaneous, and no memory effect of the previous scattering is included. We also assume the scattering is only happening in the momentum space, ie, $\vec{r}' = \vec{r}$. Then we have

$$\left.\frac{\partial f}{\partial t}\right|_{coll}(r,k) = \left.\frac{\partial f}{\partial t}\right|_{gain} - \left.\frac{\partial f}{\partial t}\right|_{loss} \quad \text{--(12c)}$$

(In the language of gain & loss of probability, the Boltzmann equation is also called the Master equation one encounters in probability theory.)

(*) In the above formalism, the thermal equilibrium condition translates into the gain & loss terms compensate each other at each k & k' states, i.e.,

$$W_{kk'}\, f_0(k)\left(1-f_0(k')\right) = W_{k'k}\, f_0(k')\left(1-f(k)\right) \quad --(13a)$$

where f_0 is the Fermi-Dirac distribution function in thermal equilibrium. This equilibrium condition, which is called the condition of detailed balance, can now be written as

$$W_{kk'}\, e^{-\beta k_B T} = W_{k'k}\, e^{-\beta \varepsilon_{k'}} . \quad ---(13b)$$

H.W. 1. For elastic scattering $W_{kk'} = W_{k'k}$. Then substitute eq(13b) on the R.H.S of eq(12) with the assumption that $f = f_0 + \delta f \approx f_0$ on the R.H.S and in the L.H.S we substitute the relaxation time approximation $\partial f/\partial t|_{coll} \approx [f_0 - (f_0+\delta f)]/\tau \approx -\delta f/\tau$. Then, show that the relaxation time can be computed as

$$\frac{1}{\tau_k} \approx \int \frac{d^3k'}{(2\pi)^3}\, W_{kk'} .$$

⊛ With the above form of the collision term, we now have to compute the scattering amplitude $W_{kk'}$ for a given potential, say $V(\vec{r})$. It is obvious that the scattering crosssection is obtained from $V(r)$ via a 2nd order perturbation theory term, but here we are not interested in the perturbation energy correction to the electron's energy, but its scattering lifetime, i.e., the imaginary part of the energy.

In this course, we are obtaining the imaginary part of the energy correction (i.e., imaginary part of the self energy) by introducing an imaginary term in the denominator (the same we did for the correlation function and the response functions in the linear response theory). (In a more regorous calculation, one goes to the complex plane and define propagator (Green's function and study its dynamics. Then both real and imaginary parts are studied in equal footings).

The 2nd order perturbation term is

$$E_k^{(2)} = \sum_{k'} \frac{|\langle k | H_{int} | k' \rangle|^2}{(E_k - E_{k'})}$$, where $H_{int} \sim V(r)$

Now we add an imaginary term $i\eta$ in the energy denominator and use the formula

$$\lim_{\eta \to 0} \frac{1}{x - i\eta} = \mathcal{P}\left(\frac{1}{x}\right) - i\pi\,\delta(x)$$, to obtain the

imaginary part, which we denote here by $W_{kk'}$ (without the k' summation:

$$W_{kk'} = -\pi \sum \left(\left| \langle k | H_{int} | k' \rangle \right|^2 \delta(E_k - E_{k'}) \right). \quad \text{---} (14)$$

$$= \frac{1}{\tau_{kk'}} \quad \text{(the definition of the relaxation time).}$$

- The scattering lifetime of a given state k is obtained by integrating over all other k' states

$$\frac{1}{\tau_k} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\tau_{kk'}} .$$

- Substituting the relaxation time in the drude formula we obtain the conductivity as

$$\sigma = e^2 \sum_{k.} \frac{n_k \tau_k}{m_k} = \sum_k \sigma_k$$

Or for the resistivity : $\frac{1}{R} = \sum_k \frac{1}{R_k}.$

Therefore, in the independent electron approximation, each k-electron states near the fermi level produce parallelly connected resistor.

- The mobility is defined by $\mu = \frac{e\tau}{m}$ which characterises the scattering process of the electrons.

(✱) We will not explicitly calculate eq (14) in this course. Some of them can however be computed easily.

H.W.

(I) For <u>electron-phonon</u> coupling case, we have already calculated τ_k.

Here. $$H_{int} = \sum_{k,q} g_q \, c^{\dagger}_{k+q} c_k \left(a_q + a^{\dagger}_{-q} \right),$$
in the previous chapter.

In this case we assume, the phonon distribution function is in thermal equilibrium. In a better, and self-consistent scheme, we have to write a Boltzmann transport equation for phonon as well and compute the relaxation time for both electron and phonon. (See Jeno Solyom, and P. Phillips)

(II) For an impurity scattering case, we have $V(r) = V\,\delta^3(r - r_0)$, with an impurity sitting at some fixed position r_0. This calculation can be simply done by going to the Fourier space of $\delta^3(r) = \frac{1}{V}\sum_q e^{i\vec{q}\cdot\vec{r}}$. Then the integration over q can be performed analytically.

(III) In the Anderson impurity model for mixed valence compound, we have such a Hamiltonian

$$H = \sum_{k,\sigma} \epsilon_k \, c^{\dagger}_{k\sigma} c_{k\sigma} + \sum_{\sigma} w_k \, a^{\dagger}_{\sigma} a_{\sigma} + \sum_k \left(V_k c^{\dagger}_{k\sigma} a_{\sigma} + h.c. \right).$$

Here the electron gets scatters to a localized electron state

(a_σ) via the last term. This results in a reduction (loss) of the valency of the electrons in its c-ionic state. This c-ion's valency then becomes fractional. Here one can compute the scattering life time for this mixed valency state.

(iv) For electron-electron scattering case, we argued in the Fermi-liquid theory that the scattering lifetime scales as $1/\tau \sim \varepsilon^2$. Calculate it.